Remark. (a) Note that it is not necessarily true that the forward image of an open set by a continuous function is open.

\[ f: \mathbb{R} \to \mathbb{R}^2 \]

\[ f(x) = \sin x \]

(b) Nor is it necessarily true that the forward image of a closed set by a continuous function is closed.

\[ f(x) = \frac{1}{x} \]

\[ f(x) = \arctan(x) \]

\[ [1, \infty) \to (0, 1] \]

\[ \mathbb{R} \to (\frac{-\pi}{2}, \frac{\pi}{2}) \]

Closed \hspace{1cm} Not open

\[ \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}) \]

Closed \hspace{1cm} Open and not closed.
**Theorem (9.3.1).** Let \( f \in C(D), f : D \to \mathbb{E}^m \). If \( D \) is compact, then \( f(D) = \{ f(x) : x \in D \} \) is compact.

\[
\text{In fact, if } f : [a, b] \to \mathbb{R} \text{ is continuous, } f([a, b]) = [c, d].
\]

**Proof:** Suppose \( D \) compact. Show \( f(D) \) is compact.

Let \( \{ \Theta \} \) be an open cover of \( f(D) \). We want to extract a finite subcover. Consider the collection \( \{ f^{-1}(\Theta) \} \). Since \( f \in C(D) \), each \( f^{-1}(\Theta) \) is relatively open in \( D \).

Hence there are open subsets of \( \mathbb{E}^n \), call them \( \{ U_\alpha \} \) such that \( f^{-1}(\Theta) = U_\alpha \cap D \). Claim: \( \{ U_\alpha \} \) is an open cover of \( D \), i.e., \( D = \bigcup_\alpha U_\alpha \). Let \( x \in D \) then \( f(x) \in f(D) \) so \( \exists \Theta_\alpha \) such that \( f(x) \in \Theta_\alpha \) so \( x \in f^{-1}(\Theta_\alpha) \subseteq U_\alpha \subseteq U \cup U_\alpha \). Since \( D \) is compact there exist \( \alpha_1, \ldots, \alpha_N \) such that \( D \subseteq \bigcup_{i=1}^N U_{\alpha_i} \).

Claim: \( f(D) \subseteq \bigcup_{i=1}^N \Theta_{\alpha_i} \). Let \( y \in f(D) \). Then for some \( x \in D, f(x) = y \). Also there is an \( \alpha_i \) such that \( x \in U_{\alpha_i} \).

Hence \( x \in \bigcup_{i=1}^N U_{\alpha_i} \cap D = f^{-1}(\Theta_{\alpha_i}) \), and so \( f(x) \in \Theta_{\alpha_i} \subseteq \bigcup_{i=1}^N \Theta_{\alpha_i} \).

So \( y \in \bigcup_{i=1}^N \Theta_{\alpha_i} \). Hence \( f(D) \) is compact.
Theorem (Extreme Value Theorem). If \( D \subseteq \mathbb{E}^n \) is compact and if \( f \in C(D) \), is a real-valued function, then \( f \) achieves its maximum and minimum values on \( D \). In other words, there exist points \( x_M \) and \( x_m \in D \) such that \( f(x) \leq f(x_M) \) and \( f(x) \geq f(x_m) \) for all \( x \in D \).

Remark: Book uses fact that a compact set is closed and bounded \( \Rightarrow f(D) \) compact \( \Rightarrow f(D) \) bounded.

\[ \exists M = \sup f(D), m = \inf f(D). f(D) \text{ closed } \Rightarrow M \text{ and } m \text{ are achieved.} \]

Proof: (1) B-W property: Since \( D \) is compact, so is \( f(D) \) hence it is bounded. Therefore \( m = \inf f(D) \) and \( M = \sup f(D) \) both exist. Will show that for some \( \tilde{x}_M \in D \), \( f(\tilde{x}_M) = M \). Since \( M = \sup f(D) \) there is a sequence \( \{y_n\} \subseteq f(D) \) such that \( y_n \to M \).

Let \( \tilde{x}_n \in D \) satisfy \( f(\tilde{x}_n) = y_n \). Since \( \tilde{x}_n \in D \), by B-W there is an \( \tilde{x}_0 \in D \) such that \( \tilde{x}_n \to \tilde{x}_0 \). For some subsequence \( \tilde{x}_{n_j} \), claim: \( f(\tilde{x}_0) = M \). Since \( f \) is continuous, \( \tilde{x}_{n_j} \to \tilde{x}_0 \) implies that \( f(\tilde{x}_{n_j}) \to f(\tilde{x}_0) \) but \( f(\tilde{x}_{n_j}) = y_{n_j} \) and \( y_{n_j} \to M \). Hence \( f(\tilde{x}_0) = M \).
(2): H-B property: Again since \( f(D) \) is bounded, 
\( m = \inf f(D) \) and \( M = \sup f(D) \) exist. Will show 
that for some \( x_M \in D \), \( f(x_M) = M \). Suppose that 
\[
\begin{array}{c}
\alpha \quad \longrightarrow \quad f \quad \longrightarrow \quad \beta \\
D
\end{array}
\] 
for no \( x \in D \), \( f(x) = M \). Then for all \( x \in D \), 
\( f(D) \neq f(\vec{x}) \). Consider 
the open sets \( \Theta_b = (-\infty, M - \frac{1}{b}) \). Claim 1: 
\( f(D) \subseteq \bigcup_{b=1}^{\infty} \Theta_b \). Claim 2: \( \exists \Theta_b \) admits no 
finite sub cover of \( f(D) \). Details are an exercise.
Theorem (9.3.3) (Open Mapping Theorem.)
Let \( f \in C(D) \), \( f : D \to \mathbb{E}^m \), \( f \) is one-to-one. If \( D \) is compact then \( f^{-1} \) is continuous.

**Rem:** (a) \( f \) is one-to-one \( \Rightarrow \) implies \( f^{-1} \) exists as a function.
(b) If \( D \) is not compact, \( f^{-1} \) need not be continuous.

**Proof:**
An idea: If we can show \( f(\text{open set}) \) is open, then this is \( (f^{-1})^{-1} \text{(open)} \) is open, so \( f^{-1} \) continuous.

\[ f: D \subseteq \mathbb{E}^n \to \mathbb{E}^m \text{ so } f: D \to f(D) \text{ } f^{-1}: f(D) \subseteq \mathbb{E}^m \to \mathbb{E}^n \]

Let \( \Theta \subseteq \mathbb{E}^n \) be open. Consider \( (f^{-1})^{-1}(\Theta) \). Want to show this is relatively open in \( f(D) \). **Claim:** \( (f^{-1})^{-1}(\Theta) = f(\Theta) \) (exercise). So must show \( f(\Theta) \) is open in \( f(D) \).

If \( \Theta \subseteq \mathbb{E}^n \) is open. Actually since \( \Theta \) need not be subset of \( D \), we are considering \( f(\Theta \cap D) \).
Actually we will show that \( f(\Omega \cap D)^c \) is relatively closed in \( f(D) \). **Claim:** \( f(\Omega \cap D)^c \) really looking at \( f(\Omega \cap D)^c \cap f(D) \).

\[
\begin{align*}
\text{Claim:} & \quad f(\Omega \cap D)^c = f(\Omega \cap D)^c = f(\Omega^c \cup D^c) \\
& = f(\Omega^c) \cap f(D) = f(\Omega^c) \cap f(D)
\end{align*}
\]

Ugh... A correct proof follows.
Proof: The idea is to show that $f^{-1}$ is continuous by showing that the inverse image under $f^{-1}$ of an open set in $E^m$ is relatively open in $f(D)$, the domain of $f^{-1}$. We will need several claims whose proofs are exercises.

Claim 1: If $f : D \to f(D)$ is one-to-one then given $A, B \subseteq D$, $f(A \cap B) = f(A) \cap f(B)$.

Claim 2: If $f : D \to f(D)$ is one-to-one then given $A \subseteq D$, $f(D - A) = f(D) - f(A)$.

Claim 3: If $f : D \to f(D)$ is one-to-one then given $A \subseteq E^m$, $(f^{-1})^{-1}(A) = f(A \cap D)$.

Let $O \subseteq E^m$ be open. Then $(f^{-1})^{-1}(O) = f(O \cap D)$. We must show that $f(O \cap D)$ is relatively open in $f(D)$. By a homework exercise it is enough to show that $f(D) - f(O \cap D)$ is relatively closed in $f(D)$. Since $O \cap D$ is relatively open in $D$, $D - (O \cap D) = O^c \cap D$ is relatively closed in $D$ and since $D$ is compact, $f(O^c \cap D)$ is compact (since $O^c \cap D$ is a closed subset of a compact set) and hence closed in $E^m$. Since $f(O^c \cap D) \subseteq f(D)$ it is a closed subset of $f(D)$. But applying our claims $f(D) - f(O \cap D) = f(D - (O \cap D)) = f(D - O) = f(O^c \cap D)$ hence $f(D) - f(O \cap D)$ is relatively closed in $f(D)$.