Looking at $\mathbb{R}^n = \{ x_1, ..., x_n : x_j \in \mathbb{R} \}$.

$\mathbb{R}^n$ is a normed linear space.

$$
\| x \| = \left( \sum_{j=1}^{n} |x_j|^2 \right)^{1/2} = \langle x, x \rangle^{1/2}.
$$

$\mathbb{R}^n$ inherits many of the topological properties of $\mathbb{R}$.

(a) $\mathbb{R}^n$ is complete. (Cauchy $\Rightarrow$ convergent)
(b) Compactness - Heine-Borel property
(c) Bolzano-Weierstrass
(d) compactness $\iff$ H-B $\iff$ B-W $\iff$ closed + bounded
(e) $\mathbb{R}^n$ has a countable dense subset, $\mathbb{Q}^n$.

Important fact: $x^k \to x$ in $\mathbb{R}^n$ $\iff$

$$
x_j^k \to x_j \text{ for all } 1 \leq j \leq n.
$$
8.2. Open Sets and Closed Sets.

A. Open Sets.

1. Definition. The open ball centered at \( a \in \mathbb{R}^n \) with radius \( r > 0 \), denoted \( B(a, r) \) is the set
\[
B(a, r) = \{ x \in \mathbb{R}^n : \| x - a \| < r \}.
\]
A set \( \mathcal{O} \subseteq \mathbb{R}^n \) is open if for each \( x \in \mathcal{O} \), there is an \( r > 0 \) such that
\[
B(x, r) \subseteq \mathcal{O}
\]

2. Examples.
   a. The empty set \( \emptyset \) is open, and \( \mathbb{R}^n \) is open.
   b. Any open ball is an open set.
   c. Any open set can be written as the union of a collection of open balls.

\[\phi \text{ is open: } \forall x \in \emptyset, \text{ anything holds.}\]

\[\mathbb{R}^n \text{ is open: Let } x \in \mathbb{R}^n, \text{ let } r = 1, B(x, 1) \subseteq \mathbb{R}^n.\]

b. Let \( B(\hat{a}, r) \) be given. Let \( \hat{x} \in B(\hat{a}, r) \). Find \( \varepsilon > 0 \) such that \( B(\hat{x}, \varepsilon) \subseteq B(\hat{a}, r) \).

   Let \( \varepsilon = r - \| \hat{x} - \hat{a} \| \), and let \( \hat{y} \in B(\hat{x}, \varepsilon) \).

   Then \( \| \hat{y} - \hat{a} \| \leq \| \hat{y} - \hat{x} \| + \| \hat{x} - \hat{a} \| < \varepsilon + \| \hat{x} - \hat{a} \| = r \)

   c. Idea: \( \mathcal{O} \subseteq \mathbb{R}^n \text{ open. } \forall x \in \mathcal{O}, \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \subseteq \mathcal{O}. \) i.e. fact \( \mathcal{O} = \bigcup_{x \in \mathcal{O}} B(x, \varepsilon_x). \)
3. **Theorem.** The union of any collection of open sets is open.

**Proof.** Let $\mathcal{O} = \{ O_x \}_{x \in A}$, a family of open sets. Show $\Theta = \bigcup O_x$ is open. Let $x \in \Theta$. Then $x \in O_x$ for some $x \in A$ so there is a $x_0 \in A$ such that $x \in O_{x_0}$. Since $O_{x_0}$ is open there is an $r > 0$ such that $B(x, r) \subseteq O_{x_0} \subseteq \Theta$. Hence $\Theta$ is open.

4. **Theorem.** The intersection of a finite number of open sets is open. The infinite intersection of open sets need not be open.

**Proof.** Let $\mathcal{O}_1, O_2 \ldots, O_n$ be open sets. Let $\Theta = \bigcap_{j=1}^{n} O_j$. Show $\Theta$ is open. Let $x \in \Theta$. Then $x \in O_j$ for all $j$. Then there are $r_j > 0$ such that $B(x, r_j) \subseteq O_j$ for all $j$. Let $r = \min_{1 \leq j \leq n} r_j$.

Then if $x \in \Theta$, $B(x, r) \subseteq B(x, r_j) \subseteq O_j$.

Hence $B(x, r) \subseteq \bigcap_{j=1}^{n} O_j = \Theta$.

Let $O_j = B(O, \frac{1}{j})$. Then $\bigcap_{j=1}^{n} O_j = \emptyset$, which is not open.
B. Closed Sets.

1. Definition. Let \( A \subseteq \mathbb{R}^n \). The point \( x \) is a \textit{limit point} of \( A \) if for every \( \varepsilon > 0 \), there is a \( y \in A \) such that \( 0 < \|x - y\| < \varepsilon \).

2. Remark.
   a. A limit point of a set \( A \) need not be an element of \( A \). For example, what are the limit points of \( B(0, 1) \)?

   b. Claim. A point \( x \) is a limit point of a set \( A \) if and only if for every \( \varepsilon > 0 \), \( B(x, \varepsilon) \cap A \) contains infinitely many points.

   **Proof.** \((\implies)\) Suppose \( x \) is a limit pt of \( A \), let \( \varepsilon > 0 \).

   Then there is a \( y \in A \) with \( 0 < \|x - y\| < \varepsilon \). Hence \( y \in B(x, \varepsilon) \cap A \). Next let \( r = \min \left\{ \frac{\|x - y\|}{2}, \varepsilon \right\} \). Since \( x \) is a limit pt of \( A \), there is \( y_1 \in A \) with \( 0 < \|x - y_1\| < r \). Let \( r_2 = \frac{\|x - y_1\|}{2} \), and continue in this fashion, we obtain a sequence \( y_h \in B(x, r) \cap A \). I claim that \( y_h \rightarrow y \) for \( h \rightarrow \infty \). Look at \( \|x - y_h\| - \|x - y_{h+1}\| \)

   Assume \( h > j \) \( \|x - y_h\| \geq \|y_j - x\| - \|y_h - y_j\| \)
But, \( 1 < v_2 < v_3 < v_4 < ... < v_j = \frac{\alpha_j - \bar{x}}{a_j} \)

Hence \( \| \frac{\alpha_j}{v_j} \| = |v_j - x| - \frac{|v_j - x|}{a_j} = \frac{|v_j - x|}{a_j} > 0 \)

(\( \Leftarrow \)) Exercise
c. **Definition.** Let $A \subseteq \mathbb{R}^n$. The point $x$ is an *isolated point* of $A$ if there exists an $r > 0$ such that $B(x, r) \cap A = \emptyset \cup \{x\}$. Note that this implies that $x \notin A$.

d. **Claim.** Every point of a set $A \subseteq \mathbb{R}$ is either an isolated point of $A$ or a limit point of $A$.

**Pl:** Exercise

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3. **Definition.** A set $F \subseteq \mathbb{R}^n$ is said to be *closed* if it contains all of its limit points.

4. **Examples.**
   a. The empty set $\emptyset$ is closed, and $\mathbb{R}$ is closed. This also shows that it is possible for a set to be both open and closed.
   b. An open ball $B(x, r)$ is not closed.
   c. Any *closed ball* $B(a, r) = \{x \in \mathbb{R}^n: \|x - a\| \leq r\}$ is closed.
   d. Any finite set is closed.

5. **Theorem.** A set is closed if and only if its complement is open.
4. (a) \( x \) is a limit pt of \( \emptyset \) means \\
\( \forall r > 0 \) there is \( y \in \emptyset \) with \( 0 < |x - y| < r \). \\
Hence the set of limit pts of \( \emptyset \) is \( \emptyset \).

(b) Look at \( B(x, r) \). Find \( y \) s.t. \\
what are the limit pts of \( B(x, r) \)?

\( \emptyset \) \\
A us: \( \exists y \in B(x, r) \) s.t. \( |x - y| \leq r \) \\
Hence \( B(x, r) \) does not contain all its limit pts.