5.24 Suppose \( \sum_{n=1}^{\infty} x_n \) converges conditionally.

This means: 1. \( \sum_{n=1}^{\infty} x_n \) converges.
2. \( \sum_{n=1}^{\infty} |x_n| \) diverges.
3. Extract 2 subsequence of \( x_n \).
   \( \{a_n\} \) subsequence of \( \geq 20 \) terms.
   \( \{b_n\} \) subsequence of \( < \) terms.

\[ y_n \text{ subseq of } x_n \text{ means } y_n = x_{\Phi(b)} \]
where \( \Phi : \mathbb{N} \to \mathbb{N} \) is one-to-one.

\( \{a_n\}, \{b_n\} \) not the same as \( x_n^+ \) and \( x_n^- \).

Know: \( \sum_{b=1}^{\infty} a_n = \infty \) and \( \sum_{b=1}^{\infty} \left[ -x_{\Phi(b)} \right] = \infty \).

Need to find a rearrangement of \( x_n, y_n \)
such that \( \sum_{n=1}^{\infty} y_n = -\infty \).

Given \( M \in \mathbb{R} \) (think \( M < 0 \)) there is an \( N \) such that for all \( n \geq N \),
\[ s_n = \sum_{b=1}^{n} y_{b} < M. \]
Idea: First let $M = -1$

Add up $n_1 + n_2 + n_3 + \ldots + n_{n_1}$ until the sum is $< -1$ and $n_1$ is the first such index.

Then add $p_1$.

Then let $M = -2$

Add up $n_1 + n_2 + \ldots + n_{n_1} + p_1 + n_{n_1+1} + \ldots + n_{n_2}$ until the sum is $< -2$.

Then add $p_2$. Continue like this.

\[ Y: n_1, n_2, \ldots, n_{n_1}, p_1, n_{n_1+1}, \ldots, n_{n_2}, p_2, \ldots, n_{n_m}, n_{n_m+1}, p_{m+1}, \ldots \]

Verify: (a) This is a rearrangement.

(b) $\sum_{n=1}^{\infty} y_n = -\infty$. $f: \mathbb{N} \rightarrow \mathbb{N}$ is one-to-one onto.
Absolute convergence: $\sum_{n=1}^{\infty} x_n$, $x_n \geq 0$.

Rahio test:

1. $\limsup_{k \to \infty} \frac{x_{k+1}}{x_k} < 1$, converges

2. $\liminf_{k \to \infty} \frac{x_{k+1}}{x_k} \geq 1$, diverges

E.g., $1, 2, \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{8}, \ldots$

Sums converge.

But $\frac{x_{k+1}}{x_k} = \begin{cases} 2 & \text{if odd} \\ \frac{1}{4} & \text{if even} \end{cases}$

So $\limsup_{k \to \infty} \frac{x_{k+1}}{x_k} = 2 > 1$

Root test works better.
2 series \( \sum_{j=1}^{\infty} x_j \quad \sum_{b=1}^{\infty} y_b \) (convergent)

We can look at \( (\sum_{j=1}^{\infty} x_j) (\sum_{b=1}^{\infty} y_b) \)

We know:

\[
(x_1 + x_2 + x_3 + x_4 + x_5) (y_1 + y_2 + y_3 + y_4) = 20 \text{ terms but a single sum.}
\]

Think about \( \sum_{j,k=1}^{\infty} x_j y_k \)

<table>
<thead>
<tr>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>X5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1</td>
<td>x1y1</td>
<td>x2y1</td>
<td>x3y1</td>
<td>x5y1</td>
</tr>
<tr>
<td>y2</td>
<td>x1y2</td>
<td>x2y2</td>
<td>x3y2</td>
<td>x5y2</td>
</tr>
<tr>
<td>y3</td>
<td>x1y3</td>
<td>x2y3</td>
<td>x3y3</td>
<td>x5y3</td>
</tr>
<tr>
<td>y4</td>
<td>[\cdot]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y5</td>
<td>[\cdot]</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There are countably many terms
So I can enumerate them

Can write as one sequence \( \sum_{n=1}^{\infty} x_n y_n \)

Issue is there are many ways to do this.

How do you make sense of \( \sum_{j,k=1}^{\infty} x_j y_k \)?
5.3. Products of Series.

**Theorem 5.3.2.** If $x_n$ and $y_n$ are absolutely summable, then so is the doubly indexed sequence $\{x_j y_k\}_{j,k=1}^\infty$ and

$$\sum_{j,k=1}^\infty x_j y_k = \left(\sum_{j=1}^\infty x_j\right) \left(\sum_{k=1}^\infty y_k\right)$$

**Proof:** Let $z_n$ be any arrangement of $\{x_j y_k\}_{j,k=1}^\infty$. Must show $\sum z_n$ converges and equals $(\sum x_j)(\sum y_k)$. Will show that $\sum |z_n|$ converges by showing sequence of partial sums $S_n = \sum |z_n|$ is bounded.

(Idea: Think about $\sum |z_n|$)
Let \( n \in \mathbb{N} \) and consider \( S_n = \sum_{b=1}^{n} |x_b| \).

Since \( x_b = X f(b) + Y g(b) \) some \( f, g : \mathbb{N} \to \mathbb{N} \).

Let \( N_1 = \max \{ f(b) : 1 \leq b \leq n \} \) and
\( N_2 = \max \{ g(b) : 1 \leq b \leq n \} \) and let
\( N_0 = \max (N_1, N_2) \). Then
\[
S_n = \sum_{b=1}^{n} |x_b| \leq \sum_{b=1}^{n} |X f(b) + Y g(b)| \leq (\sum_{j=1}^{N_0} |X_j|)(\sum_{k=1}^{N_0} |Y_k|)
\]

But since \( X_j \) and \( Y_k \) are abs summable
there is a bound \( M \) such that
\[
\sum_{j=1}^{N_0} |X_j| \text{ and } \sum_{k=1}^{N_0} |Y_k| \leq M \text{ for all } N_0.
\]

Hence \( S_n \) is bounded.

It remains to show that \( \sum_{n=1}^{\infty} S_n = (\sum_{j=1}^{\infty} X_j)(\sum_{k=1}^{\infty} Y_k) \)

(This is an exercise.)
Definition (Cauchy product)
Given sequences \( x_j, y_k \), define their \textit{Cauchy product} \( \{c_l\}_{l=2}^{\infty} \) by

\[
c_l = \sum_{j+k=l} x_j y_k = \sum_{j=1}^{l-1} x_j y_{l-j}
\]

Corollary. If \( x_j \) and \( y_k \) are absolutely summable, then

\[
\sum_{l=2}^{\infty} c_l = \sum_{j,k=1}^{\infty} x_j y_k = \left(\sum_{j=1}^{\infty} x_j\right)\left(\sum_{k=1}^{\infty} y_k\right)
\]

\textbf{Proof:} 

\[
\begin{array}{rcl}
C_2 &=& x_1 y_1 \\
C_3 &=& x_1 y_2 + x_2 y_1 \\
C_4 &=& x_3 y_1 + x_2 y_2 + x_1 y_3
\end{array}
\]

\[y_1 x_1 (y_1 x_1 y_3 x_3 y_1)\]

\[y_2 x_1 y_2 x_2 y_2 x_3 y_2\]

\[y_3 x_1 y_3 x_2 y_3 x_3 y_3\]

\[y_4 \]

\textbf{Proof:} Note that \( C_l \) is a rearrangement of \( \{x_j y_k\}_{j,k \geq 3} \). By Prev Thm, result follows.