5.7 (c) \[ |x_{n+1} - x_n| < r^n |x_1 - x_0| \]

Q: If \( |x_{n+1} - x_n| \to 0 \) is \( x_k \) Cauchy? \( \text{NO} \)

Need: \( |x_n - x_m| < \varepsilon \) if \( n, m \geq N = N(\varepsilon) \).

\[
|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m| \\
= C \left( r^{n-1} + r^{n-2} + \ldots + r^m \right) \\
= C \sum_{k=m}^{n-1} r^k \\
\text{convergent series}
\]
Unconditional convergence of \( \sum_{k=1}^{\infty} x_k \) \\
Absolute convergence of \( \sum_{k=1}^{\infty} |x_k| \).

**Rcl:** \( \sum x_k \) converges unconditionally if every rearrangement of \( x_k \) is summable.

**Q:** Is every rearrangement summable the same thing? **YES**

**Rearrangement:** \( f: \mathbb{N} \rightarrow \mathbb{N} \) bijection

\[
f(k) = y_k.
\]

Can't do: \( y: x_1, x_3, x_5, x_7, x_9, \ldots \) not \( q \)

\( x_2, x_4, x_6, \ldots, x_8, \ldots \) rearrangement
Theorem. A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if and only if it is unconditionally convergent.

Proof. ($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} |x_n|$ is absolutely convergent and let $y_k$ be a reordering of $x_k$. We will show $\sum y_k$ converges. Since $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, given $\varepsilon > 0$ there is an $N$ such that if $n, m \geq N$ then $\sum_{k=m}^{n} |x_k| < \varepsilon$.

Show that if $n, m \geq N$ then $|\sum_{k=m}^{n} y_k| < \varepsilon$, where $y_k = x_{f(k)}$ for some reordering $f$.

Choose $N_0 \geq N$ so that $f([1, N_0]) \subseteq [1, N]$, i.e., $\{f(1): 1 \leq j \leq N_0 \subseteq \mathbb{N}\} \subseteq [1, N]$. In fact, $N_0 = \max \{f(1), f(2), \ldots, f(n)\}$.

If $n, m \geq N_0$ then

$$\left| \sum_{k=m}^{n} y_k \right| \leq \sum_{k=m}^{n} |y_k| = \sum_{k=m}^{n} |x_{f(k)}| \leq \sum_{k=\min(m,N)}^{\max(n,N)} |x_k| < \varepsilon,$$

where $N_1 = \max \{f(1), f(2), \ldots, f(\min(n,N))\}$. Hence $\sum y_k$ converges.
(⇒) Suppose \( \sum_{n=1}^{\infty} x_n \) is unconditionally convergent. Show \( \sum_{n=1}^{\infty} |x_n| \) converges. Assume \( \sum_{n=1}^{\infty} |x_n| = \infty \). We will show \( \sum_{n=1}^{\infty} x_n \) is not conditionally convergent.

We can assume that \( \sum_{n=1}^{\infty} x_n \) converges, for if not the result is clear. Let

\[
x^+ = \begin{cases} x_n & \text{if } x_n \geq 0 \\ 0 & \text{if } x_n < 0 \end{cases} \quad x^- = \begin{cases} -x_n & \text{if } x_n < 0 \\ 0 & \text{if } x_n \geq 0 \end{cases}
\]

Claim: The series \( \sum_{n=1}^{\infty} x^+_n \) and \( \sum_{n=1}^{\infty} x^-_n \) both diverge.

**Proof:** Exercise

What we will show is that given \( s_0 \in \mathbb{R} \), there is a rearrangement \( y_n \) of \( x_n \) such that \( \sum_{n=1}^{\infty} y_n = s_0 \). First divide \( x_n \) into two subsequence \( x^+ \) and \( x^- \), \( x^+ \) containing the non-negative terms and \( x^- \) the negative terms. Define a rearrangement \( y_n \) as follows.

Define \( y_n \) as follows: \( y_n = p_n \) for \( n = 1, 2, \ldots, N \), where \( N \) is the first index for which \( \sum_{n=1}^{N} p_n > s_0 \).
Then define \( y_n = m_{N_2-n} \) for \( n = N_1 + 1, \ldots, N_2 \) where \( N_2 \) is the first index for which \( \sum_{n=1}^{N_2} y_n < S_0 \).

Continue in this fashion and note that at each stage only finitely many terms are required so \( y_n \) is a rearrangement of \( x_n \).

Also note that \( N_k \to \infty \) as \( k \to \infty \), and that \( N_k + 1 > N_{k+1} \). Next note that

\[
|S_0 - \sum_{n=1}^{N_2} y_n| \leq |S_0 - \sum_{n=1}^{N_k} y_n| \quad \text{where} \quad N_k \leq N < N_{k+1}
\]

\[
\leq \max(|I_{N_k}|, |P_{N_k}|)
\]

Since \( N_k \to \infty \), \( M_{N_k} \) and \( P_{N_k} \) approach zero. Therefore

\[
|S_0 - \sum_{n=1}^{N_2} y_n| \to 0 \quad \text{as} \quad k \to \infty.
\]
Now we will prove that if \( \sum x_n \) is unconditionally convergent, then for any rearrangement \( y_n \),
\[
\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n.
\]

**Proof:** Let \( S_n = \sum_{k=1}^{n} x_k \) and \( T_n = \sum_{k=1}^{n} y_k \). Show that \( |S_n - T_n| \to 0 \) as \( n \to \infty \). Let \( \varepsilon > 0 \) and find \( N \) so that if \( n \geq N \) then
\[
|\sum_{k=1}^{n} (x_k - y_k)| < \varepsilon. \]

Since both \( x_n \) and \( y_n \) are summable, \( \exists N \) such that if \( n, m \geq N \) then
\[
|\sum_{k=1}^{n} x_k| < \frac{\varepsilon}{2} \quad \text{and} \quad |\sum_{k=1}^{n} y_k| < \frac{\varepsilon}{2}.
\]

But since by previous theorem, \( x_n \) is absolutely summable, I can change above to
\[
\sum_{k=m}^{n} |x_k| < \frac{\varepsilon}{2}.
\]

As before, choose \( N_1 \) so large that
\[
\exists f(i): 1 \leq i \leq N_1 \quad \exists f(k): 1 \leq k \leq N_0 \exists f(j).
\]

\[
\begin{array}{c}
\infty \\
\text{No} \\
\text{All} \times \\
\text{N} \times \\
\end{array}
\]

\[
\begin{array}{c}
\text{As before, choose } N_1 \text{ so large that} \\
\exists f(i): 1 \leq i \leq N_1 \quad \exists f(k): 1 \leq k \leq N_0 \exists f(j),
\end{array}
\]
Then if \( n \geq N_1 \) then
\[
\sum_{k=1}^{n} (X_k - y_k) \text{ will contain terms } x_k y_k
\]
with \( k > N_0 \). If \( k \leq N_0 \) then there is a \( j \leq N_1 \) such that \( f(j) = k \) so that \( x_k = y_k \) and this term cancels out. So
\[
\left| \sum_{k=1}^{n} (X_k - y_k) \right| = \left| \sum_{k=1}^{N_0} x_k y_k + \sum_{k=N_0+1}^{n} x_k - \sum_{k=N_0+1}^{n} y_k \right| = \left| \sum_{k=1}^{N_0} x_k y_k - \sum_{k \notin f([N_1, N_0])} y_k \right|
\]
\[
\leq \left| \sum_{k=N_0+1}^{n} x_k \right| + \left| \sum_{k \notin f([N_1, N_0])} y_k \right| \leq \sum_{k=N_0+1}^{n} |x_k| < 2.
\]