5.1 Series of Constants.

Definition 5.1.1 (Convergent series) Let \( x_n \) be a sequence of numbers. The (infinite) series
\[
\sum_{n=1}^{\infty} x_n
\]
converges to \( s \) if the sequence of partial sums, \( s_n = \sum_{k=1}^{n} x_k \) converges to \( s \). In this case, we say that the sequence of terms \( x_n \) is summable and that the corresponding series is convergent. Otherwise, we say that the series is divergent.

Lemma. (Cauchy criterion.) The series \( \sum_{n=1}^{\infty} x_n \) converges if and only if the sequence of partial sums, \( s_n = \sum_{k=1}^{n} x_k \) is Cauchy, that is, given \( \varepsilon > 0 \), there is an \( N \) such that if \( n, m \geq N \) then
\[
|s_n - s_{m-1}| = |\sum_{k=m}^{n} x_k| < \varepsilon.
\]

Theorem 5.1.1 (\( n^{th} \) term test)
If \( x_n \) is a summable sequence, then \( x_n \rightarrow 0 \).

Proof. This follows immediately from the Cauchy criterion. Why? Let \( \varepsilon > 0 \), choose \( N \) in the Cauchy criterion. Then take \( n = m \geq N \). Then
\[
|s_n - s_{m-1}| = |s_n - s_{n-1}| = |x_n| < \varepsilon
\]
whenever \( n \geq N \). Hence \( x_n \rightarrow 0 \).
A sequence $x_n$ is Cauchy if $\forall \varepsilon > 0$ there is an $N$ such that $n, m \geq N$ implies $|x_n - x_m| < \varepsilon$.

Fact: $x_n$ converges iff it is Cauchy.

For series: \[ S_n = \sum_{k=1}^{n} x_k \]

\[ |S_n - S_m| = \left| \sum_{k=m+1}^{n} x_k \right| = \left| \sum_{k=m+1}^{n} x_k \right| \]

if $n > m$.

Will typically write:

\[ \sum_{k=m}^{n} x_k \rightarrow 0 \text{ as } n, m \rightarrow \infty \]

as shorthand for "$S_n$ is Cauchy".

E.g. \[ \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k \quad S_n = \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \quad \text{Guess: } S_n \rightarrow 2 \]

Look at $|S_n - 2|$. Know $S_n = \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}}$

\[ = 2 - (\frac{1}{2})^n \]

\[ |S_n - 2| = |2 - (\frac{1}{2})^n - 2| = (\frac{1}{2})^n \rightarrow 0 \text{ as } n \rightarrow \infty. \]
Theorem: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof: (Dates back to 1200s)

$$\sum_{n=1}^{8} \frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots$$

The partial sums diverge to $\infty$ but very slowly.

We have all seen: $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$
Theorem. (Abel's Formula or Summation by Parts.)
Let \(a_n\) and \(b_n\) be real-valued sequences and let
\[A_n = \sum_{k=1}^{n} a_k.\] Then for all \(n > 1\)
\[
\sum_{k=1}^{n} a_k b_k = A_n b_n - \sum_{k=1}^{n-1} A_k (b_{k+1} - b_k)
\]

Proof: \(\frac{d}{dx} (F(x)g(x)) = F(x)g'(x) + F'(x)g(x)\)

Let's mimic the proof of int by parts.
\[
\int_a^b (fg)' = \int_a^b f'g + \int_a^b g'f = f(b)g(b) - f(a)g(a)
\]
Consider sequences \(x_n, y_n\). (say \(x_0 = y_0 = 0\))
\[x_n y_n - x_{n-1} y_{n-1} = x_n y_n - x_n y_{n-1} + x_n y_{n-1} - x_{n-1} y_{n-1} = x_n (y_n - y_{n-1}) + y_{n-1} (x_n - x_{n-1})\]
\[x_n y_n = \sum_{k=1}^{n} (x_k y_k - x_{k-1} y_{k-1})
\]
\[
= \sum_{k=1}^{n} x_{k-1} (y_k - y_{k-1}) + \sum_{k=1}^{n} y_{k-1} (x_k - x_{k-1})
\]
\[
= \sum_{k=1}^{n} x_{k-1} (y_k - y_{k-1}) + \sum_{k=1}^{n} y_{k-1} (x_n - x_k)
\]
Let \( y_n = A_n = \sum_{b=1}^{n} a_b \), \( x_n = b_n \).

Note that \( y_n - y_{n-1} = a_n \). Hence we have

\[
A_nb_n = \sum_{b=1}^{n} b_nb_a + \sum_{b=1}^{n-1} A_{bn}(b_{n+1} - b_n)
\]

as required.
Theorem. (Dirichlet's Test)

Let \( a_n \) and \( b_n \) be real-valued sequences and suppose that \( s_n = \sum_{k=1}^{n} a_k \) is a bounded sequence and that \( b_n \downarrow 0 \) (that is, the sequence \( b_n \) is decreasing and converges to 0). Then \( \sum_{n=1}^{\infty} a_n b_n \) converges.

Proof: Since \( s_n \) is bounded, let's say \( |s_n| \leq B \) for all \( n \). By Abel's formula,

\[
\sum_{n=1}^{N} a_n b_n = s_N b_N - \sum_{n=1}^{N-1} s_n (b_{n+1} - b_n)
\]

Therefore,

\[
\left| \sum_{n=M}^{N} a_n b_n \right| = \left| s_N b_N - s_{M-1} b_{M-1} \right| - \sum_{n=M}^{N-1} s_n (b_{n+1} - b_n)
\]

\[
\leq \left| s_N b_N - s_{M-1} b_{M-1} \right| + \left| \sum_{n=M}^{N-1} s_n (b_{n+1} - b_n) \right|
\]

\[
\leq |s_N b_N| + |s_{M-1} b_{M-1}| + \left| \sum_{n=M}^{N-1} s_n (b_{n+1} - b_n) \right|
\]

\[
\leq B |s_N| + B |s_{M-1}| + \sum_{n=M}^{N-1} |s_n (b_{n+1} - b_n)|
\]

\[
\leq B (|b_N| + |b_{M-1}|) + B \sum_{n=M}^{N-1} |b_{n+1} - b_n|
\]

\[
\leq B (|b_N| + |b_{M-1}|) + B \sum_{n=M}^{N-1} (b_{n+1} - b_n)
\]
\[ \leq B(b_N + b_{M-1}) + B(b_N - b_M) \]

Since \( b_n \to 0 \) as \( n \to \infty \), given \( \varepsilon > 0 \), there is an \( N_0 \) such that if \( n \geq N_0 \) then \( |b_n| < \frac{\varepsilon}{4B} \). Hence if \( N, M \geq N_0 + 1 \) then

\[ |\sum_{n=M}^{N} a_n b_n| \leq B(b_N + b_{M-1}) + B(b_N - b_M) \]

\[ \leq B(b_N + b_{M-1} + b_N + b_M) \]

\[ < B\left(\frac{4\varepsilon}{4B}\right) = \varepsilon. \]

Or I could just say

\[ \leq B(b_N + b_{M-1}) + B(b_N - b_M) \to 0 \]

as \( N, M \to \infty \]

as shorthand for above.