What is an ODE (ordinary differential equation)?

- an equation involving one variable (say, $t$), a function $y(t)$, and some of its derivatives.

Some examples of ODE’s:

- example (first order ODE): \( \frac{dy}{dt} = 2y \)
- example (first order ODE): \( \frac{dy}{dt} = 3y + 4t \)
- example (second order ODE): \( \frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 7y = \cos t \)
- example (general first order ODE): \( \frac{dy}{dt} = f(t, y) \), where $f$ is a real-valued function of two real variables $t$ and $y$.

What do we mean when we refer to “the solution of an ODE”? This refers to two things:

1. a function $y(t)$ of one variable $t$;
2. an open interval $I = (a, b)$ in $\mathbb{R}$ such that both sides of the ODE agree for all $t$ in $I$ when we plug $y(t)$ in to it.
Example.

Consider the ODE \( \frac{dy}{dt} = 2y \)

1. Is \( y(t) = t^2, \ -\infty < t < \infty \) a solution?
2. Is \( y(t) = e^{2t}, \ -\infty < t < \infty \) a solution?
3. Is \( y(t) = 7e^{2t}, \ -\infty < t < \infty \) a solution?
4. Is \( y(t) = ce^{2t}, \ -\infty < t < \infty \) a solution for any choice of constant \( c \)?
Question

For the ODE $\frac{dy}{dt} = 2y$,

- what do you think all solutions look like?
- How do you know that there are no other solutions than those? Prove it.

Extension of this question

Consider the ODE $\frac{dy}{dt} = ay$, where $a$ is a fixed real number.

- What do you think all solutions look like?
- Can you show that there are no other solutions than those?
Integral curves of a first order ODE

- For the ODE \( \frac{dy}{dt} = ay \), where \( a \) is constant, we’ve seen the solution is

\[
y(t) = ce^{at}, \quad -\infty < t < \infty.
\]

- So there are infinitely many solutions, one for each choice of \( c \).
- Each such solution is called an **integral curve** of the ODE.

**Exercise**

Consider the ODE \( \frac{dy}{dt} = 2y \).

1. Plot a representative sample of integral curves.
2. What do you notice about the set of all integral curves?
3. Find the specific integral curve passing through the point \( t = 1, \ y = 6 \). Is that integral curve unique?
4. For general numbers \( t_0 \) and \( y_0 \), find the specific integral curve that passes through the point \( (t_0, y_0) \). Is that integral curve unique?
5. What general comment can you make about the set of all integral curves?

- The problem \( \frac{dy}{dt} = 2y, \ y(t_0) = y_0 \) is called an **initial value problem**.

- We showed in the above exercise that each such initial value problem (IVP) has a solution, and there is only one such solution.
More generally, for the first order ODE \( \frac{dy}{dt} = f(t, y) \), where \( f \) is a given real-valued function of two variables, each solution \( y(t) \) is called an integral curve of the ODE.

The ODE gives rise to the initial value problem

\[
\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,
\]

where \((t_0, y_0)\) is a specific point in the domain of \( f \).

We might hope that for each \((t_0, y_0)\) in the domain of \( f \), there exists an integral curve of the ODE passing through that point, and there is a unique integral curve which does this.

But we can only expect this in case \( f \) is “sufficiently nice”. We’ll discuss this in more detail in a later chapter.
Why are differential equations important?

- Say we have two variables $y$ and $t$, and we wish to study the relationship between $y$ and $t$.
- We may not know explicitly $y$ in terms of $t$, but we may know something about how $y$ changes relative to $t$.
- The derivative $\frac{dy}{dt}$ measures the rate of change of $y(t)$ with respect to $t$.
- So this information allows us to set up a differential equation in terms of $y$ and $t$.
- Using the techniques we learn about ODE's, we can then hope to say something about the functional relationship between $y$ and $t$.
- The ODE and associated IVP's provide a model which we use to describe the relationship between $y$ and $t$.
- In the next few slides we consider some examples of models. We will revisit each of them throughout the semester.
Some examples of models

Example: Population of field mice

- Let $P(t) =$ population of field mice at time $t$.
- In this model, we make the following assumptions:
  1. $P(t)$ increases at a rate proportional to its size.
  2. Mice are eaten by owls at a constant rate.
- We would like to say something about $P(t)$ for any $t$, but it’s not obvious how to do this.
- What is obvious is how to turn the above information into an ODE satisfied by $P(t)$:

$$\frac{dP}{dt} = rP(t) - k,$$

where $r$ is called the growth rate, and $k$ is called the predation rate (which is the constant rate that the owls eat the mice).

- For example, let’s take the growth rate to be $r = 0.5$/month, and take the predation rate to be $k = 15$ mice/day $= 450$ mice/month: $\frac{dP}{dt} = 0.5P - 450$.

Some Questions

- Can we predict the long-term behavior of $P(t)$, i.e. describe it for large $t$?
- Does it increase without limit?
- Does the population become extinct? Does the population oscillate?
- Could answer these questions if we could solve the ODE.
Example: Population of field mice (continued)

We describe a simple way to solve \( \frac{dP}{dt} = 0.5P - 450 \). 

- We already know how to solve \( \frac{dP_1}{dt} = 0.5P_1 \)
- namely \( P_1(t) = ce^{0.5t} \).

- Look for a constant solution, \( P_2(t) \), to the ODE: \( \frac{dP_2}{dt} = 0.5P_2 - 450 \).
- This comes out to \( P_2(t) = 900 \).
- Check to see that \( P(t) = P_1(t) + P_2(t) \) is a solution of the original ODE.
- Then we expect that the solution of the original ODE is

\[
P(t) = ce^{0.5t} + 900.
\]

- Use the solution of the ODE to plot a representative sample of integral curves of the ODE.
- Now answer the qualitative questions asked on the previous slide. Observe how those answers depend on the number of mice that are initially in the population, i.e. \( P(0) \).
Example: Newton’s law of cooling

- \( u(t) = \) temperature of an object at time \( t \).
- Place the object in a large region at constant temperature \( T \).

**Newton’s law of cooling**

The rate of change of \( u \) is proportional to the difference between \( u \) and the ambient temperature \( T \).

- This gives rise to the ODE
  \[
  \frac{du}{dt} = k(T-u),
  \]
  where \( k \) is a positive constant.

Exercise:

1. Solve the ODE (using the method described for the population model) in order to express the temperature \( u \) in terms of \( t \).
2. What do the integral curves look like?
3. What is the long-term behavior of \( u(t) \)?
Example: Mixing problems

This is an example we will return to several times in the course, with variations.

- A tank is filled with $10^6$ galons of water.
- A chemical solution containing 0.01 gm/gal of chemical enters the tank at the rate of 300 gal/hour.
- Assume the solution is instantaneously mixed.
- The solution is pumped out of the tank at the rate of 300 gal/hour.
- $q(t) = \#$ grams of chemical in the tank at time $t$ hours.

Exercise:

1. Find the IVP satisfied by $q(t)$.
2. Solve the above IVP.
3. Use the solution to describe the long-term behavior of $q(t)$.