Topology Preliminary Exam

This exam consists of 6 questions. The following notation will be used throughout this exam: \( \mathbb{R} \) denotes the set of real numbers and \( I \) denotes the closed interval \([0, 1]\). The “usual” topology on each of these will be as subspaces of \( \mathbb{R} \) with the metric \( d(x, y) = |x - y| \). Also \( S_\Omega \) denotes the well ordered ordinal space which is uncountable but for which all proper initial segments are countable. Assume all spaces are Hausdorff unless otherwise indicated.

(1) Prove that any connected open subset of \( \mathbb{R}^n \) is path connected.

(2) Let \( \{0, 1\} \) and \( \{0, 1, 2, 3\} \) be considered as discrete topological spaces. Prove that \( \{0, 1\}^\omega \) and \( \{0, 1, 2, 3\}^\omega \) are homeomorphic to each other.

(3) Prove that \( S_\Omega \) is not metrizable (so no metric on the set \( S_\Omega \) produces the same topology as the order topology).

(4) Suppose that \( X \) is a connected normal space having at least two points. Prove that the cardinality of \( X \) is at least the cardinality of \( \mathbb{R} \).

(5) Let \( [0, 1] \times [0, 1] \) be endowed with the dictionary order topology and let \( \pi_1 \) and \( \pi_2 \) denote the projection functions to the first and second coordinates. Suppose \( f : I \to [0, 1] \times [0, 1] \) is continuous. Prove that \( \pi_1(f(I)) \) is a constant map.

(6) Prove that a compact Hausdorff space must be normal.