This exam consists of 5 questions.

(1) Let $\mathbb{R}$ denote the set of real numbers, and for $x, y \in \mathbb{R}$, $d(x, y) = |x - y|$. Then we know that $(\mathbb{R}, d)$ is a complete metric space with the “usual” metric. Show that completeness is not preserved by homeomorphism, by finding a non-complete metric space $(M, d^*)$ homeomorphic to $(\mathbb{R}, d)$ and an onto homeomorphism, $h : \mathbb{R} \to M$.

(2) (a) Let $C[0, 1]$ denote the space of all continuous real-valued functions on $[0, 1]$ equipped with the maximum norm $\|f\|_\infty = \max\{|f(x)| : x \in [0, 1]\}$. Show that $(C[0, 1], \|\cdot\|_\infty)$ is not a Hilbert space, i.e., show that it is impossible to define a scalar product $(\cdot, \cdot)$ on $C[0, 1]$ such that $\|f\|_\infty = (f, f)^{1/2}$ for all $f \in C[0, 1]$.

(b) Define a scalar product on $C[0, 1]$ (equipped with the usual addition and scalar multiplication of functions) and show that with this scalar product, $C[0, 1]$ becomes a Euclidean space. Then show that the so-defined space is not complete.

(3) Let $M$ be a finite-dimensional subspace of a normed linear space $X$. Show that there is a closed subspace $N \subset X$ with $X = M \oplus N$. (Hint: Given a basis $x_1, \ldots, x_n$ of $M$, find $f_1, \ldots, f_n \in X^*$ with $f_i(x_j) = \delta_{i,j}$.)

(4) Let $X = C[1, 4]$ be the space of real–valued continuous functions on $[1, 4]$ equipped with the maximum norm, $\|f\|_\infty = \max\{|f(x)| : x \in [1, 4]\}$. For $f \in X$, define

$$T(f) = \int_1^2 f(x) \, dx - \int_3^4 f(x) \, dx$$

and $S(f) = f(2)$. Determine whether $S$ and $T$ are continuous. If a functional is continuous, find its norm; if not explain why not.

(5) Let $K(x, y)$ be a fixed function of two variables, continuous on the square $[0, 1] \times [0, 1]$, and let $A \in \mathcal{L}(L^2[0, 1], L^2[0, 1])$ be the operator defined by

$$Af(x) = \int_0^1 K(x, y) f(y) \, dy.$$ 

Prove that if $\{f_n\}_{n=1}^\infty$ satisfies $\|f_n\|_2 \leq 1$ for all $n = 1, 2, \ldots$, then the set $\{Af_n\}_{n=1}^\infty$ is equicontinuous. This means that given $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in [0, 1]$ are such that $|x - y| < \delta$, then $|Af_n(x) - Af_n(y)| < \epsilon$ for all $n = 1, 2, \ldots$. 
