Algebra Preliminary Exam

This exam consists of 5 questions.

(1) Let $G$ be a group and $N$ a normal subgroup of $G$. Let $G/N = \{gN : g \in G\}$ be the set of cosets. Show that $G/N$ is a group under the operation $gN \cdot g'N = gg'N$.

(2) Let $G$ be an abelian group and $n \in \mathbb{N}$. Let $G_n = \{g \in G : g^n = e\}$ where $e \in G$ is the identity element of $G$ and let $G^n = \{g^n : g \in G\}$. Show that $G/G_n \cong G^n$.

(3) Let $p$ be a prime number.
   (a) Show that the center of a finite $p$-group is nontrivial.
   (b) Show that any group of order $p^2$ is abelian.

   Prove any lemmas you use.

(4) Let $\phi : R \to R'$ be a ring homomorphism. Show that $\ker(\phi)$ is an ideal of $R$ and that the image $\phi(R)$ is a subring of $R'$.

(5) Let $\alpha$ be an irrational real root of the polynomial $X^2 + bX + c$ where $b$ and $c$ are rational. Show that $\mathbb{Q}[\alpha]$, the smallest subring of $\mathbb{R}$ containing the rationals $\mathbb{Q}$ and $\alpha$, is a field.