FACTS. Let $S$ be a set of real numbers. Then

(i) $\sup S = \infty$ iff $S$ is not bounded above, i.e. for any number $M > 0$, there exists $s \in S$ such that $s > M$.

(ii) $\sup S = a$ where $a \in \mathbb{R}$ iff $a$ is an upper bound for $S$ and for every $\varepsilon > 0$, there exists $s \in S$ such that $s > a - \varepsilon$.

"If (i) is just the definition of $\sup S = \infty$.

$(\Rightarrow)$ By def., $\sup S = a$ is an upper bound for $S$.

For every $\varepsilon > 0$, since $a$ is the LEAST upper bound, and $a - \varepsilon < a$, $a - \varepsilon$ cannot be a upper bound for $S$. So there exists $s \in S$ such that $s > a - \varepsilon$.

$(\Leftarrow)$ Suppose $a$ is an upper bound for $S$ and $a$ satisfies the condition. If $b < a$, let $b < a$, and let $\varepsilon = a - b > 0$. There exists $s \in S$ such that $s > a - \varepsilon = b$. So $b$ cannot be an upper bound for $S$. So $a$ is the LEAST upper bound.