Riemann Integrals

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In what follows, \( \mu \) denotes the Lebesgue measure.

**Definition 0.1** Let \( I = [a, b] \) be a closed interval in \( \mathbb{R} \). A finite ordered subset \( a_0 < a_1 < \cdots < a_n \) of \( I \) with \( a_0 = a, a_n = b \) is called a partition \( P \) of \( I \). The norm of \( P \), denoted by \( \| P \| \), is the maximum of \( a_i - a_{i-1}, i = 1, \ldots, n \).

**Definition 0.2** Let \( I = [a, b] \) be a closed interval in \( \mathbb{R} \) and \( f : I \to \mathbb{R} \). \( f \) is said to be Riemann integrable on \( I \) if there exists a real number \( A \) satisfying the following condition: For every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for any partition \( P : a_0 < \cdots < a_n \) with \( \| P \| < \delta \) and any points \( x_i, i = 1, \ldots, n \) with \( a_{i-1} \leq x_i \leq a_i \), one has

\[
|\sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) - A| < \epsilon.
\]

The number \( A \), denoted by \( (R) \int_{a}^{b} f(x) \, dx \), is called the Riemann integral of \( f \) on \( [a, b] \).

**Proposition 0.1** Every Riemann integrable function on a closed interval \( I \) is bounded on \( I \).

**Proof.**
If \( f \) is unbounded on \( I \), then for any partition \( P : a_0 < \cdots < a_n \), \( f \) will be unbounded on one of the intervals \( [a_{i-1}, a_i], i = 1, \ldots, n \). Then for any real number \( A, | \sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) - A| \) can be made arbitrarily large by appropriate choices of \( x_i, i = 1, \ldots, n \). Hence \( f \) cannot be Riemann integrable.

Let \( f \) be a bounded real-valued function defined on a closed interval \( I = [a, b] \). If \( P : a = a_0 < a_1 < \cdots < a_n = b \) is a partition of \( I \), define

\[
u(P, f) = \sum_{i=1}^{n} u_i \chi_{[a_{i-1}, a_i]}
\]
and

\[
l(P, f) = \sum_{i=1}^{n} l_i \chi_{[a_{i-1}, a_i]}
\]
Proof.

Let $\epsilon > 0$ be such that for any partition $P : a_0 < \cdots < a_n$ with $\|P\| < \delta$ and any points $x_i, i = 1, \ldots, n$ with $a_{i-1} \leq x_i \leq a_i$, one has

$$| \sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) - A | < \frac{\epsilon}{4}.$$ 

Now let $P : a_0 < \cdots < a_n$ be a partition with $\|P\| < \delta$. Choose points $x_i, y_i \in [a_{i-1}, a_i]$ such that $u_i - x_i < \frac{\epsilon}{4M}$ and $y_i - l_i < \frac{\epsilon}{4M}$. It then follows easily from the triangle inequality that $U(P, f) - L(P, f) < \epsilon$.

$(3) \Rightarrow (2)$: This follows immediately from $0 \leq U(f) - L(f) \leq U(P, f) - L(P, f)$ for any partition $P$.

$(2) \Rightarrow (3)$: If (b) holds, then for every $\epsilon > 0$, there exist partitions $P$ and $Q$ such that $U(P, f) - L(Q, f) < \epsilon$. Let $T = P \vee Q$. Then $U(T, f) - L(T, f) \leq U(P, f) - L(Q, f) < \epsilon$.

$(3)$ and $(2) \Rightarrow (1)$: Let $A = U(f) = L(f)$. Let $\epsilon > 0$. Choose a partition $P$ such that $U(P, f) - L(P, f) < \epsilon/2$. Let $\delta_i$ be the minimum lengths of the intervals determined by $P$ and let $\delta = \min\{\delta_i, \frac{\epsilon}{2M}\}$, where $n$ is the number of intervals determined by $P$ and $M \neq 0$ is a bound for $|f(x)|, x \in I$. Let $Q : a = a_0 < \cdots < a_m = b$ be a partition of $I$ with $\|Q\| < \delta$ and $a_{i-1} \leq x_i \leq a_i, i = 1, \ldots, m$. Let $r$ be the Riemann sum $\sum_{i=1}^{m} f(x_i)(a_i - a_{i-1})$. Let $T = P \vee Q$. Each point in $P$ is either an endpoint of an interval determined by $Q$ or belongs to the interior of exactly one interval determined by $Q$. It follows that we need to choose at most $n-2$ points in addition to $\{x_i : i = 1, \cdots, m\}$ for the choice of points for a Riemann sum $s$ for $T$. And $|r - s| \leq (n-2)M\delta < \epsilon/2$. It follows that

$$r - \frac{\epsilon}{2} - A \leq s - A \leq U(T, f) - A \leq U(P, f) - A \leq U(P, f) - L(P, f) < \frac{\epsilon}{2}$$

and

$$A - r - \frac{\epsilon}{2} \leq A - s \leq A - L(T, f) \leq A - L(P, f) \leq U(P, f) - L(P, f) < \frac{\epsilon}{2}$$

Lemma 0.1 Let $f$ be a bounded real-valued function defined on a closed interval $I = [a, b]$. The following are equivalent:

1. $f$ is Riemann integrable on $I$.
2. $U(f) = L(f)$.
3. For every $\epsilon > 0$, there exists a partition $P$ of $I$ such that $U(P, f) - L(P, f) < \epsilon$.

Proof.

$(1) \Rightarrow (3)$: Suppose $f$ is Riemann integrable on $I$ and let $A$ be its Riemann integral. Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that for any partition $P : a_0 < \cdots < a_n$ with $\|P\| < \delta$ and any points $x_i, i = 1, \ldots, n$ with $a_{i-1} \leq x_i \leq a_i$, one has

$$| \sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) - A | < \frac{\epsilon}{4}.$$
showing that $|r - A| < \epsilon$. ■

**Corollary 0.1** Let $f$ be a Riemann integrable function on $[a, b]$. Then for any partitions $P$ and $Q$, $L(P, f) \leq (R) \int_{a}^{b} f(x) \, dx \leq U(Q, f)$ and $U(f) = L(f) = (R) \int_{a}^{b} f(x) \, dx$.

**Lemma 0.2** Let $f$ be a Riemann integrable function on $I = [a, b]$. If $P_n$ is a sequence of partitions of $I$ with $\|P_n\| \to 0$ as $n \to \infty$, then $\lim_n U(P_n, f) \to (R) \int_{a}^{b} f(x) \, dx$ and $\lim_n L(P_n, f) \to (R) \int_{a}^{b} f(x) \, dx$.

**Proof.** Let $A = (R) \int_{a}^{b} f(x) \, dx$. Let $\epsilon > 0$. There exists $\delta > 0$ such that for any partition $P : a_0 < \cdots < a_k$ with $\|P\| < \delta$ and any points $x_i, i = 1, \ldots, k$ with $a_{i-1} \leq x_i \leq a_i$, one has

$$|\sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) - A| < \frac{\epsilon}{2}$$

For such partition $P$, if one chooses $a_{i-1} \leq x_i \leq a_i$ such that $u_i - f(x_i) \frac{\epsilon}{2\|P\|}$, where $u_i = \sup\{f(x) : x \in [a_{i-1}, a_i]\}$, then $U(P, f) - \sum_{i=1}^{n} f(x_i)(a_i - a_{i-1}) < \frac{\epsilon}{2}$.

So $U(P, f) - A < \epsilon$. The proof for $L(P, f)$ is similar. ■

**Proposition 0.2** If $f$ is Riemann integrable on $I = [a, b]$, then it is also Lebesgue integrable on $I$ and the two integrals are equal.

**Proof.** Let $P_n$ be the partition of $[a, b]$ with each interval having length $\frac{b - a}{2^n}$. Using the notations preceding Lemma ??, let $u_n = u(P_n, f)$ and $l_n = l(P_n, f)$. Then $u_n$ and $l_n$ are bounded monotonic functions. Let $u$ and $l$ be the limits of $u_n$ and $l_n$ respectively. By Bounded Convergence Theorem and the preceding lemma, both $u$ and $l$ are Lebesgue integrable and $\int u \, d\mu = \int l \, d\mu = (R) \int_{a}^{b} f(x) \, dx$. Thus $\int u - l \, d\mu = 0$. Since $u \geq f \geq l$, we have $u - l = 0$ a.e. and hence $u = f = v$ a.e. Hence $f$ is Lebesgue integrable and $\int f \, d\mu = \int u \, d\mu = (R) \int_{a}^{b} f(x) \, dx$. ■

Let $X$ be a topological space and $f$ be a function from $X$ to $\mathbb{R}$. For each $x \in X$, the oscillation $\omega(x, f)$ of $f$ at $x$ is defined to be

$$\inf\{\text{diam } f(U) : U \text{ a neighborhood of } x\}$$

**Proposition 0.3** The function $\omega(x, f)$ is upper semicontinuous. $f$ is continuous at $x$ if and only if $\omega(x, f) = 0$.

**Theorem 0.1** A real function $f$ on a closed interval $I$ is Riemann integrable if and only if it is bounded on $I$ and is continuous a.e. on $I$.
Proof.
Suppose \( f \) is Riemann integrable on \( I \). By Proposition 0.1, \( f \) is bounded. Let \( m \) be a positive integer and \( E_m = \{ x : \omega(x, f) \geq 1/m \} \). We need to show that \( \mu(E_m) = 0 \). Let \( \epsilon > 0 \) be given. By Lemma 0.1 there exists a partition \( P \) of \( I \) such that
\[
U(P, f) - L(P, f) < \frac{\epsilon}{m}.
\]

If an interval \( I_i \) determined by \( P \) contains some points of \( E_m \) in its interior, then \( u_i - l_i \geq 1/m \). It follows that if \( J_1, \ldots, J_k \) are such intervals determined by \( P \) then
\[
\frac{\epsilon}{m} > U(P, f) - L(P, f) \geq \frac{1}{m}(|J_1| + \cdots + |J_k|)
\]
and \( |J_1 + \cdots + |J_k| < \epsilon \). So \( E_m \) is a union of a subset of the finite set \( \{ a = a_0, a_1, \ldots, a_n = b \} \) and a set of measure \( < \epsilon \); so \( \mu(E_m) < \epsilon \). Since \( \epsilon > 0 \) is arbitrary, one has \( \mu(E_m) = 0 \). Since the set \( E = \{ x : f \text{ discontinuous at } x \} = \bigcup_{m=1}^{\infty} E_m \), one obtains \( \mu(E) = 0 \).

Next assume that \( f \) is bounded on \( I = [a, b] \) and is continuous a.e. on \( I \). Let \( \epsilon > 0 \) be given. Let \( m \) be a positive integer such that \( \frac{b-a}{m} < \frac{\epsilon}{4} \). Since the set \( E_m = \{ x : x \in I, \omega(x, f) \geq 1/m \} \) is of measure 0 there exist disjoint open intervals \( I_i, i = 1, 2, \ldots \) in \( \mathbb{R} \) such that \( E_m = \bigcup_{i=1}^{\infty} I_i \) and \( \sum_{i=1}^{\infty} |I_i| < \frac{\epsilon}{16m} \), where \( \mu \) is a bound for \( |f(x)|, x \in I \). Since \( E_m \) is compact (it is a closed subset of \( I \) by the upper semicontinuity of \( \omega \)), we may assume that \( E_m \subset \bigcup_{i=1}^{\infty} I_i \). \( I \setminus \bigcup_{i=1}^{\infty} I_i \) is a disjoint union of a finite number of closed subintervals \( J_1, \ldots, J_k \) of \( I \). Let \( J \) be one of these subinterval. For each \( x \in J \) since \( \omega(x, f) < 1/m \) there exists an open interval \( O_x \) containing \( x \) such that \( u - l < 1/m \) where \( u = \sup \{ f(y) : y \in O_x \cap J \} \) and \( l = \inf \{ f(y) : y \in O_x \cap J \} \). By the compactnes of \( J \), a finite number of these open intervals \( O_x \)‘s cover \( J \). The endpoints of these finite number of \( O_x \)’s together with those of \( J \) make a partition of \( J \). We do this for each \( J_i \)’s. Now these partitions together with endpoints of \( I_i, i = 1, \ldots, n \) and \( I \) make up a partition \( P \) for \( I \). It is quite obvious now that \( U(P, f) - L(P, f) < \epsilon \). Since \( \epsilon > 0 \) is arbitrary, \( f \) is Riemann integrable on \( I \) by Proposition ??.

Example 0.1 Show that the functions
\[
f(x) = \begin{cases} 
x & x \text{ irrational} \\ p \sin \frac{1}{q} & x \text{ rational }, x = \frac{p}{q} \text{ in its lowest terms} \end{cases}
\]
and
\[
g(x) = \begin{cases} 
0 & x = 0 \\ \sin \frac{1}{x} & 0 < x \leq 1 
\end{cases}
\]
are Riemann integrable on \([0, 1]\).

Example 0.2 Show that the function
\[
h(x) = \begin{cases} 
0 & x \text{ rational} \\ 1 & x \text{ irrational} 
\end{cases}
\]
is Lebesgue integrable but not Riemann integrable on \([0, 1]\).