#1 or #7. a, c, d only. Note that \( f(A \cap B) \subseteq f(A) \cap f(B) \) and generally equality fails. It is easy to find an example where \( A \cap B = \emptyset \) and \( f(A) \cap f(B) = \emptyset \).

#2 or #8. False. Not all infinite sets are of the same cardinality.

#3 or #9. b, c only. If a set is not open, it need not be closed.

#4 or #10. It is closed.

#5 or #11. It is neither open nor closed.

#6 or #12. It is 1. The largest subsequential limit is 1.

#7 or #1. f is cont. at 0 only.

#8 or #2. \( \dim X + \dim Y = \dim (X \oplus Y) = \dim X + \dim Y \).

#9 or #3. If \( a \leq x \) then \( a = x \). Note that a maximal element need not be maximum. For example,

\[
\begin{array}{c}
\alpha \\
\delta \\
\epsilon
\end{array}
\]

\( \alpha \) is maximal but not maximum by \( \delta \neq \beta \) (\( \alpha, \beta \) are incomparable).

#10 or #4. True.

#11 or #5. \( \text{rank } T + \text{nullity } T = \dim V \) for \( T : V \rightarrow W \).

#12 or #6. \( \text{rank } T = \text{rank } T^T \).

#13. A is a Hamel basis of \( V \) if it is a maximal linearly independent set.

#14. \( \text{rank } T = \text{dimension of } T(V) \) (the range of \( T \))

\( \text{nullity } T = \text{dimension of the null space of } T \)
# 15.

**T:** \( \mathbb{R}^n \to \mathbb{R}^{n*} \)

\[ x \mapsto Tx \]

\[ Tx(y) = x_1y_1 + \cdots + x_ny_n \]

**T is linear b/c**

\[ T(u+v)(y) = (u+v)y_1 + \cdots + (u+v)y_n \]

\[ = u_1y_1 + \cdots + u_ny_n + v_1y_1 + \cdots + v_ny_n \]

\[ = Tu(y) + Tv(y) \]

\[ = (Tu + Tv)(y) \quad \text{for all } y \in \mathbb{R}^n \]

\[ \Rightarrow \quad T(u+v) = Tu + Tv \]

\[ T(\lambda u)(y) = \lambda u_1y_1 + \cdots + \lambda u_ny_n \]

\[ = \lambda (u_1y_1 + \cdots + u_ny_n) \]

\[ = \lambda Tu(y) \]

\[ = (\lambda Tu)(y) \quad \text{for all } y \in \mathbb{R}^n \]

\[ \therefore \quad T(\lambda u) = \lambda Tu. \]

Since \( \dim \mathbb{R}^n = \dim \mathbb{R}^{n*} \)

**T is one-to-one b/c** \( Tu = 0 \)

\[ \Rightarrow \quad Tu(y) = 0 \quad \forall y \]

\[ \Rightarrow \quad u_1y_1 + \cdots + u_ny_n = 0 \quad \forall y \in \mathbb{R}^n \]

\[ \Rightarrow \quad u_1 = 0, \ldots, u_n = 0 \]

Since \( \dim \mathbb{R}^n = n = \dim \mathbb{R}^{n*} \), **T is an isomorphism** (you don't need to show that it is onto: one-to-one \( \Rightarrow \) onto).

# 16.

**T:** \( V \to W \)

\[ T^*: W^* \to V^* \]

\[ T^*(w^*) = w^* \circ T \]

# 17.

(8). \( A \in \mathcal{C} \Rightarrow A' \in \mathcal{C} \)

(9). \( A_i \in \mathcal{C} \quad \forall i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{C} \).
18. (b) \( \mu \left( \bigcup_{i=1}^{w} A_i \right) = \sum_{i=1}^{w} \mu(A_i) \) if \( A_i \in \mathcal{A} \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \).

19. Since a point singleton \( \{x\} \) is closed, every singleton is a Borel set. The set of rationals, being a countable union of singletons, is thus a Borel set. Therefore, the set of irrationals, being the complement of \( \mathbb{Q} \), is a Borel set.

20. Let \( P : V \to V \) be a projection, i.e., \( P^2 = P \).

For every \( v \in V \), \( P(v - Pv) = P(v - P^2v) = P(v - Pv) = 0 \)
\[ \implies v - Pv \in \text{null}(P) \]
\[ \implies v = Pv + (v - Pv) \in \text{R}(P) + \text{N}(P) \].

If \( y \in \text{R}(P) \cap \text{N}(P) \), then
\[ y = Pw \text{ for some } w \in V \]
and \( Py = P(Pw) = P^2w = Pw = y \).

But \( y \in \text{N}(P) \implies Py = 0 \implies y = Py = 0 \).

So \( \text{R}(P) \cap \text{N}(P) = \{0\} \).

\[ \therefore V = \text{R}(P) \oplus \text{N}(P) \).

Note: It is not true for a general \( T : V \to V \) that \( \text{R}(T) \oplus \text{N}(T) = V \). For example, \( T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \).

\( \text{R}(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \).
\( \text{N}(T) = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \) (check!)

and \( V \neq \text{R}(T) \oplus \text{N}(T) \neq \{0\} \).