Exercise 1.14 2.

Prove that \( \#A < \#\mathcal{P}(A) \).

\[ P: \text{ Clearly } \#A \leq \#\mathcal{P}(A) \text{ since } a \mapsto \{ a \} \text{ is an injection from } A \text{ into } \mathcal{P}(A). \text{ We shall show that } A \not\sim \mathcal{P}(A). \]

Suppose on the contrary that \( A \sim \mathcal{P}(A) \) and let \( f: A \to \mathcal{P}(A) \) be an bijection. Let

\[ S = \{ x \in A : x \notin f(x) \} . \]

\( S \) is a subset of \( A \) (possibly empty). Since \( f \) is onto, there exists \( y \in A \) s.t. \( f(y) = S \). Either \( y \in S \) or \( y \notin S \).

If \( y \in S \), then \( y \notin f(y) = S \Rightarrow y \notin S \) by the definition of \( S \), a contradiction.

If \( y \notin S \), then \( y \notin f(y) \) (by the def. of \( S \)), but then \( y \in S \) since \( S = f(y) \), a contradiction.

Hence \( A \not\sim \mathcal{P}(A) \) i.e. \( \#A \neq \#\mathcal{P}(A) \). Consequently \( \#A < \#\mathcal{P}(A) \).

Proof of \( \mathbb{R} \sim \mathcal{P}(\mathbb{N}) \).

First we prove that \( \mathcal{P}(\mathbb{N}) \sim [0,1) \) using Cantor-Bernstein theorem.

We need to find an injection from \( \mathcal{P}(\mathbb{N}) \) into \([0,1)\) and an injection from \([0,1)\) into \( \mathcal{P}(\mathbb{N}) \).

Let \( S \in \mathcal{P}(\mathbb{N}) \). Define \( f(S) = 0.a_1a_2...a_n... \), the decimal number 0.\( a_1a_2...a_n... \), where \( a_i = 0 \) if \( i \notin S \) and \( a_i = 1 \) if \( i \in S \). \( f \) is clearly an injection since the numbers in \([0,1)\) can have different decimal expansion only when one of its decimal expansion is of the form 0.\( a_1a_2...a_n... \) where \( a_n = 9 \) for all \( n \geq n_0 \) (for a fixed \( n_0 \in \mathbb{N} \)).
So we have an injection from $\mathcal{P}(\mathbb{N})$ into $[0, 1)$.

For each $x \in [0, 1)$, $x$ has a unique \underline{binary} expansion $0.a_1a_2\ldots$ which does not end with all 1's. Define $g(x)$ to be the subset $q$ of $\mathbb{N}$:

$$g(x) = \{ i \in \mathbb{N}: a_i = 1 \}$$

$g$ is an injection from $[0, 1)$ into $\mathcal{P}(\mathbb{N})$.

By Cantor-Bernstein theorem, $\mathcal{P}(\mathbb{N}) \sim [0, 1)$.\]

Next we show that $[0, 1) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$. Again, we use Cantor-Bernstein theorem.

Let $a, b$ be two numbers in $(-\frac{\pi}{2}, \frac{\pi}{2})$ satisfying

Let $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Define

$$f(x) = \frac{1}{2\pi} (\pi - a) x + a \quad 0 \leq x < 1$$

Then $f$ is an injection from $[0, 1)$ into $(-\frac{\pi}{2}, \frac{\pi}{2})$. On the other hand, the map $g$ defined by

$$g(x) = \frac{\pi}{\pi} (x + \frac{\pi}{2})$$

is an injection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ into $[0, 1)$. Hence $[0, 1) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$.

Lastly the map $\tan: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ is a bijection from $\mathbb{R}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Hence $\mathbb{R} \sim (-\frac{\pi}{2}, \frac{\pi}{2})$.

Therefore $\mathbb{R} \sim [0, 1) \sim \mathcal{P}(\mathbb{N}) \Rightarrow \mathbb{R} \sim \mathcal{P}(\mathbb{N})$.\]