Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 2:20 pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) A portion of the graph of a function defined on the interval $[-5, 5]$ is shown. Complete the graph assuming that the function is (a) even (b) odd. Label each.

2. (15 points) (a) If $f$ is a one-to-one function and $f(x)$ is never zero, show that $g(x) = 1/f(x)$ is also one-to-one.

   If $g(a) = g(b)$ then $\frac{1}{f(a)} = \frac{1}{f(b)}$ which implies $f(a) = f(b)$.

   But $f$ is one-to-one, so $a = b$. This shows $g$ is one-to-one.

(b) Assuming (a) is true even if you had trouble showing it fully, this shows that $g$ has an inverse function. When the graph of $f(x)$ only for $1 \leq x \leq 5$ is the line through the points $(1, 8)$ and $(5, 20)$, find a formula for the inverse function of $g(x)$ and describe its domain, given the restriction $1 \leq x \leq 5$ for the domain of $f$.

   \[ f(x) = 8 + m \cdot (x - 1) \quad \text{[point-slope]} \quad \text{and} \quad m = \frac{20 - 8}{5 - 1} = \frac{12}{4} = 3 \quad \text{so} \quad f(x) = 3(x - 1) + 8 = 3x + 5 \]

   \[ g(x) = \frac{1}{f(x)} = \frac{1}{3x + 5} \quad \text{and for} \quad 1 \leq x \leq 5, \ \frac{1}{g(x)} = \frac{1}{8} \quad \text{so} \quad g^{-1}(x) = \frac{1}{8} (x - 5) \]

   \[ g^{-1} \text{ maps } \left[ \frac{1}{20}, \frac{1}{8} \right] \text{ onto } [1, 5] \text{ to find its formula, solve} \]

   \[ y = g(x) = \frac{1}{3x + 5} \quad \text{for} \quad x \text{ in terms of} \quad y: \quad \frac{1}{y} = 3x + 5, \quad 3x = \frac{1}{y} - 5 \]

   \[ x = \frac{1}{3} \left( \frac{1}{y} - 5 \right) \]
3. (10 points) If the function \( y = \left(\frac{1}{3}\right) \cdot 2^x \) is written as an exponential in base \( e \) (meaning in the form \( y_0 e^{kx} \)), what are the numbers \( y_0 \) and \( k \)?

\[
y_0 e^{kx} = \frac{1}{3} \cdot 2^x \quad \text{forces} \quad y_0 = \frac{1}{3} \quad \text{and} \quad e^{kx} = 2^x
\]

[for example, use \( x = 0 \) or take \( \ln \) of both sides]. Then \( e^{kx} \) must equal \( 2^x \) so \( k = \ln(2) \). Check \( \frac{1}{3} e^{(\ln2)x} = \frac{1}{3} \cdot 2^x \)

Alternate solution: take \( \ln \) of both:

\[
\ln(y_0 e^{kx}) = \ln(y_0) + \ln(e^{kx}) = \ln(y_0) + kx
\]

\[
\ln(\frac{1}{3} \cdot 2^x) = \ln(\frac{1}{3}) + \ln(2^x) = \ln(\frac{1}{3}) + (\ln 2) \cdot x
\]

4. (15 points) A ball of pizza dough dropped from the Leaning Tower of Pisa falls a distance \( y = 16t^2 \) feet after \( t \) seconds.

Describe algebraically and graphically the average speed of the pizza dough over the following intervals: from 2 seconds to 3 seconds; from 2 seconds to \( t \) seconds. Then find the instantaneous speed at 2 seconds. **YOU MUST SHOW SOME WORK FOR INSTANTANEOUS SPEED.**

\[
y(2) = 16 \cdot 4 = 64 \text{ ft} , \quad y(3) = 16 \cdot 9 = 144 \text{ ft} \quad \text{so}
\]

\[
\text{avg speed, 2 sec to 3 sec} = \frac{144 - 64}{3 - 2} = \frac{80}{1} = 80 \text{ ft/sec}
\]

\[
\text{avg speed, 2 sec to } t \text{ sec} = \frac{16t^2 - 64}{t - 2} = 16t + 32 = 16(t + 2)
\]

\[
\text{limit of avg speed, 2-sec to } t \text{ sec, as } t \to 2 \text{ sec}
\]

\[
\text{instantaneous speed} = \lim_{{t \to 2}} 16t + 32 = 64 \text{ ft/sec}
\]
5. (10 points) Describe how the radian measure of an angle is defined. Then describe the trigonometric functions $\cos \theta$ and $\sin \theta$ in terms of the radian measure and the unit circle.

Radian measure = distance around unit circle \[ \text{[fraction of circumference (2\pi)]} \]

$(\cos \theta, \sin \theta)$ represent $(x, y)$ coordinates on circle.

6. (15 points) Find the limits:

(a) \[ \lim_{x \to 2} \frac{x^2 + 4}{x + 4} = \lim_{x \to 2} \frac{x^2 + 4}{x + 4} = \frac{4 + 4}{2 + 4} = \frac{8}{6} = \frac{4}{3} \]

(b) \[ \lim_{y \to 1} \sqrt{4y^2 + 4y + 1} = \sqrt{\lim_{y \to 1} 4y^2 + 4y + 1} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3 \]

(c) \[ \lim_{x \to 3} \frac{x^2 - 9}{x + 3} \]

Can't do \( \frac{0}{0} \) so divide then conquer

\[ \lim_{x \to 3} \frac{x^2 - 9}{x + 3} = \lim_{x \to -3} \frac{x - 3}{x + 3} = -3 - 3 = -6 \]
7. (10 points) For the function \( f(t) \) graphed below, find the following limits or explain why they do not exist:

(a) \( \lim_{t \to 0^-} f(t) = 0 \) [from graph near 0 on left]

(b) \( \lim_{t \to 0^+} f(t) = 0 \) [from graph near 0 on right]

(c) \( \lim_{t \to -1} f(t) \text{ DOES NOT EXIST - Left limit is 0, Right limit is 1} \)

8. (15 points) (a) Give a formal definition of limit of a function \( f \):

\[ \lim_{x \to x_0} f(x) = L \text{ as } x \to x_0; \]

Given any \( \varepsilon > 0 \), there is a \( \delta \) such that for all \( 0 < |x - x_0| < \delta \), \( |f(x) - L| < \varepsilon \)

(b) For the function \( f(x) = \frac{1}{x} \), find a \( \delta > 0 \) such that for all \( x \) satisfying \( 0 < |x - 1| < \delta \), the inequality \( |f(x) - 1| < \frac{1}{2} \) holds.

\[ \left| \frac{1}{x} - 1 \right| < \frac{1}{2} \iff \frac{1}{2} < \frac{1}{x} < \frac{3}{2} \iff 2 > x > \frac{2}{3} \]

This forces \( \delta \leq \frac{1}{3} \) [shorter distance from 1 to edges]

(c) Explain part (b) graphically.