1. (1.6, # 30) Let $V = M_{2\times2}(F)$ and let $W_1$ and $W_2$ be the subsets of $V$ as defined.

**Proof:** We show that $W_1$ is a subspace. Let $X, Y \in W_1$. Then say

$$X = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} e & f \\ g & e \end{pmatrix}$$

Then $X + Y = \begin{pmatrix} a + e & b + f \\ c + g & a + e \end{pmatrix}$ and since the entries on the main diagonal are equal, this matrix is in $W_1$. Similarly it is easy to see that $\alpha A$ is in $W_1$ for $\alpha \in F$. Hence $W_1$ is a subspace. The same thing works for $W_2$.

To find a basis for $W_1$, we note that $a, b$ and $c$ are free variables. So here is a candidate for a basis:

$$\left\{ u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$  

Clearly $\begin{pmatrix} a & b \\ c & a \end{pmatrix} = au_1 + bu_2 + cu_3$. So the set spans $W_1$. To check linear independence, one writes:

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{pmatrix} = 0.$$  

Clearly this implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus $\{u_1, u_2, u_3\}$ is a linearly independent set. Hence it is a basis. Thus $\dim(W_1) = 3$.

It is also direct to see that

$$\left\{ v_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

is a basis of $W_2$. Thus $W_2$ has dimension 2 (there are two free variables in describing elements of $W_2$).

Since $W_1 + W_2$ is bigger than $W_1$, it follows that $\dim(W_1 + W_2) > 3$. But this is a subspace of $V$ which has dimension 4. Hence $W_1 + W_2$ has dimension 4. Finally to describe the intersection of the two spaces, notice that for a matrix to be in the intersection entire on the main diagonal must be equal to be in $W_1$, while they must be 0 to also be in $W_2$. Therefore we have

$$W_1 \cap W_2 = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : a, b \in F \right\}.$$  

Clearly there is only one free variable. Hence the space has dimension 1.
2. (Sec. 2.1, # 14a) The next solution is included just to show you how it is done. Let \( T : V \to W \) be linear. Show that \( T \) is injective iff \( T \) carries linearly independent sets to linearly independent sets.

**Proof:** Let \( T \) be injective. Let \( u_1, \ldots, u_m \) be a linearly independent set in \( V \). We want to show that \( T(u_1), \ldots, T(u_m) \) is a linearly independent set (in \( W \)). Suppose

\[
a_1 T(u_1) + \ldots + a_m T(u_m) = 0
\]

Then by the linearity of \( T \) we have \( T(a_1 u_1 + \ldots + a_m u_m) = 0 \). But \( T \) is injective, hence \( a_1 u_1 + \ldots + a_m u_m = 0 \). But this original set is linearly independent. Hence all \( a_i = 0 \) and we are done in this direction.

Suppose that \( T \) carries linearly independent sets to linearly independent sets. Show that \( T \) is injective. It suffices to show that if \( u \) is a nonzero vector in \( V \), then \( T(u) \neq 0 \). But any set with one nonzero vector is a linearly independent set. Thus by assumption, \( \{T(u)\} \) is a linearly independent set and so \( T(u) \neq 0 \). This proves that the null space of \( T \) is trivial.

(Sec. 2.1, # 14b) Suppose that \( T \) is one-to-one and that \( S \) is a subset of \( V \). Prove that \( S \) is linearly independent iff \( T(S) \) is linearly independent.

**Proof:** Suppose that \( T \) is one-to-one. Let \( S = \{u_1, u_2, \ldots, u_n\} \) be a linearly independent set in \( V \). Then by part (a) above, \( T(S) \) is linearly independent. Conversely, assume that \( T(S) \) is linearly independent. We want to show that \( S \) is linearly independent (we don’t need the fact that \( T \) is one-to-one for this). Suppose that

\[
a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0.
\]

Then \( 0 = T(a_1 u_1 + a_2 u_2 + \ldots + a_n u_n) = a_1 T(u_1) + a_2 T(u_2) + \ldots + a_n T(u_n) \). But by assumption the set \( \{T(u_i)\}_{i=1,2,\ldots,n} \) is a linearly independent set. Hence \( a_1 = a_2 = \ldots = a_n = 0 \). Thus \( S \) is a linearly independent set.

3. (Sec. 2.1, # 17) Let \( V \) and \( W \) be finite dimensional vector spaces and let \( T : V \to W \) be linear.

(a) Prove that if \( \dim(V) < \dim(W) \), then \( T \) cannot be onto

(b) Prove that if \( \dim(V) > \dim(W) \), then \( T \) cannot be one-to-one.

**Proof:** The result we will use for both parts is Theorem 2.3 which states that \( \dim(V) = \text{nullity}(T) + \text{rank}(T) \). Also note that for any transformation \( T \), it is onto iff \( \text{rank}(T) = \dim(W) \) and it is one-to-one iff \( \text{nullity}(T) = 0 \).

(a) Suppose that \( \dim(V) < \dim(W) \). Note that by Theorem 2.3, \( \text{rank}(T) \leq \dim(V) \) for any linear transformation, while by assumption \( \dim(V) < \dim(W) \). Hence \( \text{rank}(T) < \dim(W) \) and so \( T \) is not onto.

(b) By the theorem we have \( \dim(V) = \text{nullity}(T) + \text{rank}(T) \) or \( \dim(V) - \text{nullity}(T) = \text{rank}(T) \). Since the range space is a subspace of \( W \), we have that \( \text{rank}(T) \leq \dim(W) < \dim(V) \). Thus \( \text{nullity}(T) > 0 \) and so \( T \) is not one-to-one.