Multiresolution Analysis.

A new look at the Haar system.

**Definition.** For each \( j \in \mathbb{Z} \), define the *approximation operator* \( P_j \) on \( L^2(\mathbb{R}) \), by

\[
P_j f(x) = \sum_k \langle f, p_{j,k} \rangle p_{j,k}(x).
\]

Define the *approximation space* \( V_j \) by

\[
V_j = \text{span}\{p_{j,k}(x)\}_{k \in \mathbb{Z}}.
\]

Since \( \{p_{j,k}(x): k \in \mathbb{Z}\} \) is an orthonormal system on \( \mathbb{R} \), \( P_j f(x) \) is the function in \( V_j \) best approximating \( f(x) \) in the \( L^2 \) sense.

Define the *detail operator* \( Q_j \) on \( L^2(\mathbb{R}) \), by

\[
Q_j f(x) = P_{j+1} f(x) - P_j f(x).
\]

Define the *wavelet space* \( W_j \) by

\[
W_j = \text{span}\{h_{j,k}(x)\}_{k \in \mathbb{Z}}.
\]

Since \( \{h_{j,k}(x)\}_{k \in \mathbb{Z}} \) is an orthonormal system on \( \mathbb{R} \), \( Q_j f(x) \) is the function in \( W_j \) best approximating \( f(x) \) in the \( L^2 \) sense.
**Theorem. (a)** The scale $J$ Haar system on $\mathbb{R}$ is a complete orthonormal system on $\mathbb{R}$. (The scale $J$ Haar system is

$$\{p_{J,k}(x), h_{j,k}(x): j \geq J; k \in \mathbb{Z}\}.$$ 

(b) The Haar system is a complete orthonormal system on $\mathbb{R}$. (The Haar system is

$$\{h_{j,k}(x): j k \in \mathbb{Z}\}).$$

Proving that the Haar system is a complete orthonormal system on $\mathbb{R}$ amounts to showing the following.

**Theorem. (a)** \(\lim_{j \to \infty} \|P_j f - f\|_2 = 0\), and

(b) \(\lim_{j \to \infty} \|P_j f\|_2 = 0\).

(c) Given \(f \in C^0_c(\mathbb{R})\),

$$Q_j f(x) = \sum_k \langle f, h_{j,k} \rangle h_{j,k}(x).$$
**Definition.** A *multiresolution analysis* on $\mathbb{R}$ is a sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$ satisfying:

(a) For all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$.

(b) $\overline{\text{span}}\{V_j\}_{j \in \mathbb{Z}} = L^2(\mathbb{R})$. That is, given $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, there is a $j \in \mathbb{Z}$ and a function $g(x) \in V_j$ such that $\|f - g\|_2 < \epsilon$.

(c) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$.

(d) A function $f(x) \in V_0$ if and only if $D_{2j}f(x) \in V_j$.

(e) There exists a function $\varphi(x)$, $L^2$ on $\mathbb{R}$, called the *scaling function* such that the collection $\{T_n\varphi(x)\}$ is an orthonormal system of translates and

$$V_0 = \overline{\text{span}}\{T_n\varphi(x)\}.$$
Examples of MRA.

Note: In order to define an MRA it is sufficient to either (1) specify $V_0$ then show that there is a scaling function $\varphi(x)$ such that $V_0 = \text{span}\{T_n\varphi\}$, or (2) specify the scaling function $\varphi(x)$ and define $V_0 = \text{span}\{T_n\varphi\}$.

(a) The Haar MRA. $\varphi(x) = p_{0,0}(x) = 1_{[0,1]}(x)$.

(b) The Bandlimited MRA. $V_0$ is the set of all functions $f$ bandlimited to $[-1/2, 1/2]$. 
(c) The Meyer MRA.
Given \( k \in \mathbb{N} \) (or \( k = \infty \)), a function \( b(x) \) is a \( C^k \) bell function over \([-1/2, 1/2]\) provided that \( b(x) \) is \( C^k \) on \( \mathbb{R} \) and satisfies the following conditions:

(a) \( b(x) = 1 \) if \( |x| \leq 1/3 \),
(b) \( b(x) = 0 \) if \( |x| > 2/3 \),
(c) \( 0 \leq b(x) \leq 1 \) for all \( x \in \mathbb{R} \), and
(d) \( \sum_n |b(x + n)|^2 \equiv 1 \).

Now take \( \varphi(x) \) to be the inverse Fourier transform of a \( C^k \) bell-function.
(d) **The Piecewise Linear MRA.** Let $V_0$ consist of all functions $f \in L^2(\mathbb{R}) \cap C^0(\mathbb{R})$ linear on the intervals $I_{0,k}$, for $k \in \mathbb{Z}$. Think of this as a stepped-up version of the Haar MRA.

Define the function $\varphi(x) = (1 - |x|) 1_{[-1,1]}(x)$.

**Lemma.** If $f \in V_0$ then $f(x) = \sum_n f(n) T_n \varphi(x)$ pointwise and in $L^2(\mathbb{R})$.

**Lemma.** $V_0 = \overline{\text{span}} T_n \varphi$.

**Theorem.** There is a function $\tilde{\varphi}(x)$, $L^2$ on $\mathbb{R}$, such that:
(a) $\{ T_n \tilde{\varphi}(x) \}$ is an orthonormal system of translates, and
(b) $V_0 = \overline{\text{span}} \{ T_n \tilde{\varphi}(x) \}$. 
Some results about collections of the form \( \{T_{n}g\}_{n \in \mathbb{Z}} \).

(a) If \( \{T_{n}g\}_{n \in \mathbb{Z}} \) is an orthonormal system on \( \mathbb{R} \), then \( f \in \overline{\text{span}}T_{n}g \) if and only if

\[
    f(x) = \sum_{n} \langle f, T_{n}g \rangle T_{n}g(x)
\]

in \( L^{2} \) if and only if there is a Fourier series \( \hat{c}(\gamma) \) with period 1 such that

\[
    \hat{f}(\gamma) = \hat{g}(\gamma) \hat{c}(\gamma).
\]

(b) The collection \( \{T_{n}g(x)\} \) is an orthonormal system of translates if and only if for all \( \gamma \in \mathbb{R} \),

\[
    \sum_{n} |\hat{g}(\gamma + n)|^{2} = 1.
\]

(c) If for some \( 0 < A < B \)

\[
    A \leq \sum_{n} |\hat{g}(\gamma + n)|^{2} \leq B
\]

then there is a function \( \tilde{g} \in L^{2}(\mathbb{R}) \), such that:

(i) \( \{T_{n}\tilde{g}(x)\} \) is an orthonormal system of translates and

(ii) \( \overline{\text{span}}\{T_{n}g(x)\} = \overline{\text{span}}\{T_{n}\tilde{g}(x)\} \).
Wavelet basis from MRA

**Theorem.** (The two-scale relation) There exists \( \{h(k)\} \in \ell^2 \) such that

\[
\varphi(x) = \sum_k h(k) 2^{1/2} \varphi(2x - k)
\]

in \( L^2 \) on \( \mathbb{R} \). Moreover, we may write

\[
\hat{\varphi}(\gamma) = m_0(\gamma/2) \hat{\varphi}(\gamma/2),
\]

where

\[
m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k) e^{-2\pi ik\gamma}.
\]
Theorem. (The wavelet ”recipe”) Let \( \{V_j\} \) be an MRA with scaling function \( \varphi(x) \) and scaling filter \( h(k) \). Define the wavelet filter \( g(k) \) by

\[
g(k) = (-1)^k h(1 - k)
\]

and the wavelet \( \psi(x) \) by

\[
\psi(x) = \sum_k g(k) 2^{1/2} \varphi(2x - k).
\]

Then

\[
\{\psi_{j,k}(x)\}_{j,k \in \mathbb{Z}}
\]

is a wavelet orthonormal basis on \( \mathbb{R} \).

Alternatively, given any \( J \in \mathbb{Z} \),

\[
\{\varphi_{J,k}(x)\}_{k \in \mathbb{Z}} \cup \{\psi_{j,k}(x)\}_{j,k \in \mathbb{Z}}
\]

is an orthonormal basis on \( \mathbb{R} \).

Remark. Taking the Fourier transform gives that

\[
\hat{\psi}(\gamma) = m_1(\gamma/2) \hat{\varphi}(\gamma/2),
\]

where

\[
m_1(\gamma) = e^{-2\pi i (\gamma + 1/2)} m_0(\gamma + 1/2),
\]
(a) The Haar wavelet. In this case, we can compute the scaling and wavelet filters directly.

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) = \frac{1}{\sqrt{2}} \varphi_{1,0}(x) + \frac{1}{\sqrt{2}} \varphi_{1,1}(x).$$

Therefore,

$$h(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1, \\ 0 & \text{if } n \neq 0, 1, \end{cases}$$

Therefore,

$$g(n) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, \\ -\frac{1}{\sqrt{2}} & \text{if } n = 1, \\ 0 & \text{if } n \neq 0, 1. \end{cases}$$

and

$$\psi(x) = \frac{1}{\sqrt{2}} \varphi_{1,0}(x) - \frac{1}{\sqrt{2}} \varphi_{1,1}(x)$$

$$= \varphi(2x) - \varphi(2x-1)$$

$$= 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x).$$
(b) The Bandlimited wavelet. Here it is more convenient to work on the transform side. Recall that $\hat{\varphi}(\gamma) = 1_{[-1/2,1/2]}(\gamma)$. Since $\hat{\varphi}(\gamma/2) = 1_{[-1,1]}(\gamma)$, it follows that

$$\varphi(\gamma) = m_0(\gamma/2) \varphi(\gamma/2),$$

where $m_0(\gamma)$ is the period 1 extension of $1_{[-1/4,1/4]}(\gamma)$.

Thus, $m_1(\gamma)$ is the period 1 extension of the function

$$e^{-2\pi i (\gamma+1/2)} (1_{[-1/2,-1/4]}(\gamma) + 1_{[1/4,1/2]}(\gamma))$$

so that

$$\hat{\psi}(\gamma) = m_1(\gamma/2) \hat{\varphi}(\gamma/2)$$

$$= -e^{-\pi i \gamma} (1_{[-1,-1/2]}(\gamma) + 1_{[1/2,1]}(\gamma)).$$

By taking the inverse Fourier transform,

$$\psi(x) = \frac{\sin(2\pi x) - \cos(\pi x)}{\pi(x - 1/2)}$$

$$= \frac{\sin \pi(x - 1/2)}{\pi(x - 1/2)} (1 - 2 \sin \pi x).$$
(c) The Meyer wavelet. Recall that

$$\hat{\varphi}(\gamma) = \begin{cases} 
0 & \text{if } |\gamma| \geq 2/3, \\
1 & \text{if } |\gamma| \leq 1/3, \\
s(\gamma + 1/2) & \text{if } \gamma \in (1/3, 2/3), \\
c(\gamma - 1/2) & \text{if } \gamma \in (-2/3, -1/3), 
\end{cases}$$

Therefore, $$\hat{\varphi}(\gamma) = m_0(\gamma/2) \hat{\varphi}(\gamma/2),$$ where $$m_0(\gamma)$$ is the period 1 extension of the function $$\hat{\varphi}(2\gamma) 1_{[-1/2,1/2]}(\gamma).$$

$$\psi(x)$$ is defined by

$$\hat{\psi}(\gamma) = -e^{-\pi i \gamma} m_0(\gamma/2 + 1/2) \hat{\varphi}(\gamma/2)$$

and

$$\hat{\psi}(\gamma) = \begin{cases} 
0 & \text{if } |\gamma| \leq 1/3 \text{ or } |\gamma| \geq 4/3, \\
s(\gamma - 1/2) & \text{if } \gamma \in (1/3, 2/3], \\
c(\gamma/2 - 1/2) & \text{if } \gamma \in (2/3, 4/3), \\
s(\gamma/2 + 1/2) & \text{if } \gamma \in (-4/3, -2/3), \\
c(\gamma + 1/2) & \text{if } \gamma \in [-2/3, -1/3). 
\end{cases}$$
(d) The Piecewise Linear wavelet. Recall that

\[ \hat{\phi}(\gamma) = \hat{\phi}(\gamma) \Phi(\gamma) = \frac{\sqrt{3} \hat{\phi}(\gamma)}{(1 + 2 \cos^2(\pi \gamma))^{1/2}}, \]

where \( \varphi(x) = (1 - |x|) 1_{[-1,1]}(x) \) and

\[ \Phi(\gamma) = (\sum_n |\hat{\varphi}(\gamma + n)|^2)^{-1/2}. \]

Also,

\[ \hat{\varphi}(\gamma) = \cos^2(\pi \gamma/2) \varphi(\gamma/2). \]

Therefore,

\[ \hat{\varphi}(\gamma) = \cos^2(\pi \gamma/2) \left( \frac{1 + 2 \cos^2(\pi \gamma/2)}{1 + 2 \cos^2(\pi \gamma)} \right)^{1/2} \hat{\varphi}(\gamma/2), \]

so that

\[ m_0(\gamma) = \cos^2(\pi \gamma) \left( \frac{1 + 2 \cos^2(\pi \gamma)}{1 + 2 \cos^2(2\pi \gamma)} \right)^{1/2}. \]
Therefore,

\[ m_1(\gamma) = -e^{-2\pi i \gamma} \sin^2(\pi \gamma) \left( \frac{1 + 2 \sin^2(\pi \gamma)}{1 + 2 \cos^2(2\pi \gamma)} \right)^{1/2}. \]

and

\[ \hat{\psi}(\gamma) = d(\gamma/2) \hat{\varphi}(\gamma/2). \]

where

\[ d(\gamma) = -\sqrt{3} e^{-\pi i \gamma} \sin^2(\pi \gamma/2) \]

\[ \times \left( \frac{1 + 2 \sin^2(\pi \gamma)}{(1 + 2 \cos^2(2\pi \gamma))(1 + 2 \cos^2(\pi \gamma))} \right)^{1/2} \]

Therefore

\[ \psi(x) = \sum_n d(n) \varphi_{1,n}(x), \]

where \( d(n) \) is the \( n^{th} \) Fourier coefficient of \( d(\gamma) \).