3.2 Systems of first order linear ODE's

- The general system of first order linear ODE's is given by

\[
\begin{align*}
\frac{dx_1}{dt} &= P_{11}(t)x_1(t) + P_{12}(t)x_2(t) + g_1(t) \\
\frac{dx_2}{dt} &= P_{21}(t)x_1(t) + P_{22}(t)x_2(t) + g_2(t),
\end{align*}
\]

where the six functions \( P_{11}(t), P_{12}(t), P_{21}(t), P_{22}(t), g_1(t), g_2(t) \) are given, and the aim is to say something about each of the unknown functions \( x_1(t) \) and \( x_2(t) \).

- This is more elegantly written using matrix notation and matrix arithmetic as follows:

\[
\frac{d\vec{x}}{dt} = P(t) \vec{x}(t) + \vec{g}(t),
\]

where the unknown vector valued function \( \vec{x}(t) \) is defined to be

\[
\vec{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]

and the matrices

\[
P(t) := \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}, \quad \vec{g}(t) := \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}
\]

consist of given functions.

- If \( \vec{g}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), we say that it is a **homogeneous system**. The reason is that in this case the constant function \( \vec{x}(t) \equiv \vec{0} \) is a solution.

- If the matrix-valued function \( P(t) \) consists of all constant functions, we say that the system has **constant coefficients**.
Exercise.

For the initial value problem

\[ \vec{x}'(t) = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 2t \\ -3 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \]

verify that the vector-valued function

\[ \vec{x}(t) = e^{-t} \begin{bmatrix} 2t - 1 \\ t - 1 \end{bmatrix} + \begin{bmatrix} 6t + 2 \\ 2t - 1 \end{bmatrix} \]

is a solution.
We write some terminology associated with the system \( \frac{d\vec{x}}{dt} = P(t) \vec{x}(t) + \vec{g}(t) \).

\( x_1 \) and \( x_2 \) are called the **state variables**, and \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is called the **state vector**.

The \( x_1x_2 \)-plane is called the **phase plane**.

If \( \vec{x}(t) \) is a specific solution of the system, then \( t \mapsto \vec{x}(t) \) is a curve in the phase plane, and we call it a **trajectory** of the system.

A trajectory is determined once we prescribe an initial condition on the solution of the system, such as prescribing the value of \( \vec{x}(0) \).

A representative sample of trajectories drawn in the phase plane is called a **phase portrait** of the system.

The **qualitative theory** consists of drawing the phase portrait of the system.

For the constant coefficient homogeneous case, the system is

\[
\vec{x}'(t) = A\vec{x}(t)
\]

where \( A \) is a constant \( 2 \times 2 \) matrix. We shall see that the phase portrait is entirely determined by the eigenvalue-eigenvector pairs of the coefficient matrix \( A \).
Let’s compare the autonomous ODE \( \frac{dy}{dt} = -3y \) with the system \( \frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \vec{x}(t). \)

\( \frac{dy}{dt} = -3y: \)

- The phase line is

- The black dot represents the equilibrium solution \( y \equiv 0. \)
- The two arrows in the phase line denote the curves \( t \mapsto y(t) \) as they approach the equilibrium solution, so they are trajectories.
- So the “phase line” is also the phase portrait of the ODE \( \frac{dy}{dt} = -3y \)

\( \frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \vec{x}: \)

- The equilibrium solution is the constant solution \( \vec{x} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). So it corresponds to the constant trajectory \( t \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), which is a single point in the phase plane, namely the origin.
- The phase portrait tells us how the nonconstant trajectories relate to the equilibrium point at the origin of the phase plane.
- But two dimensional curves are more diverse than one dimensional curves, so we can expect this to be more involved than our study of phase lines for autonomous ODE’s.
Consider the general linear initial value problem:

\[ \frac{d\vec{x}}{dt} = P(t)\vec{x}(t) + \vec{g}(t), \quad \vec{x}(t_0) = \vec{x}_0, \]

where the four functions that make up \( P(t) \), the two functions that make up \( \vec{g}(t) \), the number \( t_0 \), and the two numbers that make up \( \vec{x}_0 \) are all given.

Since the system is linear, we should expect a strong existence and uniqueness theorem. We state it next, although the proof is far beyond the scope of the course.

**Theorem (Existence and uniqueness for initial value problems for linear systems of ODE’s)**

Suppose the functions which make up \( P(t) \) and \( \vec{g}(t) \) are all continuous on an open interval \( I \) containing the number \( t_0 \). Then the above system has a unique solution \( \vec{x}(t) \) with domain all of \( I \).
Direction Field of a system

- For the general first order linear system \( \frac{d\vec{x}}{dt} = P(t)\vec{x}(t) + \vec{g}(t) \), the \textbf{direction field} consists of associating the vector \( \frac{d\vec{x}}{dt} \) to each point \( \vec{x}(t) \) in the phase plane.

- A representative sample of those vectors gives the picture of the direction field.

- By drawing curves in the direction of the arrows, we get the phase portrait of the system.

\[ \text{Exercise.} \]

For the system \( \frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{x}, \) draw a few of the arrows which make up the direction field.

- This is done more easily using Mathematica.
Exercise.

In order to see some of the variety of things which can occur for two-dimensional linear systems, sketch direction fields and phase portraits for each of the following linear homogeneous systems using mathematica.

(a) \[ \vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} \]

(b) \[ \vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x} \]

(c) \[ \vec{x}' = \begin{bmatrix} 1/2 & -5/4 \\ 2 & -1/2 \end{bmatrix} \vec{x} \]

(d) \[ \vec{x}' = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \vec{x} \]

We’ll see that the diverse phenomena we see in these pictures can be completely explained using eigenvalues and eigenvectors.