A MIHALISIN - KLEE THEOREM FOR FANS

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Abstract. The Mihalisin - Klee Theorem states that an orientation of a 3-polytopal graph is induced by an affine function on some 3-polytope realizing the graph if the orientation is acyclic, has a unique source and a unique sink, and admits three independent monotone paths from the source to the sink. We replace the requirement that the orientation is acyclic with the assumption that it has no directed cycle contained in a face of the orientation, and show that such orientations are induced by 3-dimensional fans.

1. Introduction

A graph G is called d-polytopal if there exists a d-dimensional polytope P so that the graph formed by the vertices and edges of P is isomorphic to G. The classical theorem of Steinitz [9] states that a graph is 3-polytopal if and only if it is 3-connected and planar. We apply the term d-polytopal to an *orientation* of a d-polytopal graph G if there is a d-dimensional polytope P and an affine function on P so the graph of P is isomorphic to G and the orientation of the graph agrees with the directions of increase of the affine function. Holt and Klee [5] showed that if an orientation of the graph of a d-polytope is d-polytopal, then there are d independent monotone paths from the source to the sink. The Mihalisin - Klee Theorem is a converse to the Holt - Klee Theorem for the case $d = 3$ (For examples that show that no such theorem is possible in dimension 4, see [4] and [8].)

Theorem 1.1. *(Mihalisin and Klee* [8]*) Suppose that K is an orientation of a 3-polytopal graph G. Then the digraph K is 3-polytopal if it is acyclic, has a unique source and a unique sink, and admits three independent monotone paths from source to sink.*

The characterization of Mihalisin and Klee was used by Kaibel et. al. [6] to analyze the expected behavior of randomized simplex algorithms on linear programs with three variables.

A *fan* in $\mathbb{R}^d$ is a collection $\mathcal{F} = \{C_1, C_2, \ldots, C_t\}$ of nonempty polyhedral cones, such that

1) Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.

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2) The intersection of any two cones in $\mathcal{F}$ is a face of both.

A fan $\mathcal{F}$ is *complete* if the union of its cones is $\mathbb{R}^d$. It is *pointed* if the zero vector is one of its cones. All of the fans discussed in this paper will be assumed to be complete and pointed. We will also assume that all of the cones in $\mathcal{F}$ are distinct.

For more background on fans and polytopes, see [2] and [11]. The *dual graph* of a complete pointed d-fan $\mathcal{F}$ is the graph $G_{\mathcal{F}}$ that has a vertex for every d-dimensional cone of $\mathcal{F}$ and has an edge between two vertices if the corresponding cones share a $(d-1)$-dimensional face. We have a function μ from the set of vertices and edges of $G_{\mathcal{F}}$ to the set of $d-$ and $(d-1)-$dimensional cones of $\mathcal{F}$ that is an inclusion-reversing bijection. If $\mathcal{F}$ is a complete pointed d-fan and $g \in \mathbb{R}^d$, we say that g is *generic* with respect to $\mathcal{F}$ if it is not in the linear span of any $(d-1)$-dimensional cone of $\mathcal{F}$. For $\mathcal{F}$ and generic g one can define the digraph $D_{\mathcal{F},g}$ which has underlying undirected graph $G_{\mathcal{F}}$ and directs any edge $\{v, w\}$ from v to w if $\mu(w)$ is on the same side as g of $\mu(\{v, w\})$.

The digraph $D_{\mathcal{F},g}$ was studied for complete simplicial fans by Kleinschmidt and Onn [7]. It is clear that $D_{\mathcal{F},g}$ has a unique source and a unique sink, namely the vertices corresponding to the cones containing $-g$ and g. If $\mathcal{F}$ is the normal fan of a polytope, then $G_{\mathcal{F}}$ is the graph of the polytope and $D_{\mathcal{F},g}$ orients edges in the direction of increase of the function $g^T x$. The theorem of Holt and Klee states that $D_{\mathcal{F},g}$ has d disjoint paths from the source to the sink if $\mathcal{F}$ is the normal fan of a polytope. The following generalization was proved in [4].

Theorem 1.2. *Let $\mathcal{F}$ be a complete pointed d-fan, and let $g \in \mathbb{R}^d$ be generic with respect to $\mathcal{F}$. Then there are d independent paths from the source to the sink of $D_{\mathcal{F},g}$.*

Our goal is to weaken the acyclicity condition in the Mihalisin - Klee Theorem and show that one gets digraphs $D_{\mathcal{F},g}$ for 3-fans. In order to state the weaker condition, we need the following fact, proved by Whitney [10]:

Proposition 1.3. *The face lattice of a 3-polytope is determined by its graph.*

This means that every realization of a given 3-polytopal graph has the same vertex sets defining 2-dimensional faces of the polytope. We can therefore refer to circuits of a 3-polytopal graph as either face boundaries or separating circuits. Now we can state our theorem:

Theorem 1.4. *Suppose that K is an orientation of a 3-polytopal graph G. If none of the face boundaries are directed cycles, it has a unique source and unique sink, and it admits three independent monotone paths from source to sink, then the digraph K is $D_{\mathcal{F},g}$ for some fan $\mathcal{F}$ and generic g.*

We follow the proof of Mihalisin and Klee (some aspects of which were inspired by a proof of Barnette and Grünbaum [1]) to some extent. However, the acyclicity is crucial to building the polytope in their proof, and we will not be able to make use of that.

In Figure 1., assume that the origin O is in the interior of the prism on the left. For each nonempty face of the prism, the fan $\mathcal{F}$ has a cone generated by the rays from O to the vertices of the face. We assume that the vector g is pointing directly upward. Unfortunately, this means that it is not generic with respect to $\mathcal{F}$, because it is contained in the span of the cones $cone(\{OA, OD\}), cone(\{OC, OE\})$, and $cone(\{OB, OF\})$. The graph $G_{\mathcal{F}}$ is drawn to the right, with some edges directed. For example, $\mu(G) = cone(\{OA, OC, OD, OE\})$ and $\mu(J) = cone(\{OA, OB, OC\}$. We have directed the edge from G to J because the cone $\mu(G)$ and g are on opposite sides of $cone(\{OA, OC\})$. If we twist slightly the top face of the prism, then the resulting vertex sets corresponding to the quadrilateral faces of the prism will no longer be coplanar. The cones generated by these sets will, however, still be pointed convex cones with four extreme rays. After the twist, the vector g will be generic with respect to $\mathcal{F}$ and the edges $\{G, I\}, \{I, H\}, \{H, G\}$ form a cycle in $D_{\mathcal{F},g}$ that is directed oppositely to the twist.

2. Properties of the digraph

Suppose that K is a directed graph. A *path* in K is a sequence of distinct vertices $(x_0, x_1, \ldots, x_k)$ such that for $i = 1, 2, \ldots, k$, vertices x_{i-1} and x_i are adjacent in the graph underlying K. The path is *monotone* if for $i = 1, 2, \ldots, k$ the edge containing x_{i-1} and x_i is oriented from x_{i-1} to x_i. It is called *antitone* if the path $(x_k, x_{k-1}, \ldots, x_0)$ is monotone. A *directed cycle* has the same definition as a monotone path except that $x_0 = x_k$. A set of monotone paths from a vertex x to a vertex y of K will be called *independent* if the only vertices that appear in more than one of the paths are x and y.

We assume for the rest of this section that K is an orientation of a 3-polytopal graph. If K has no directed cycle in a face boundary, has a unique source x and a unique sink

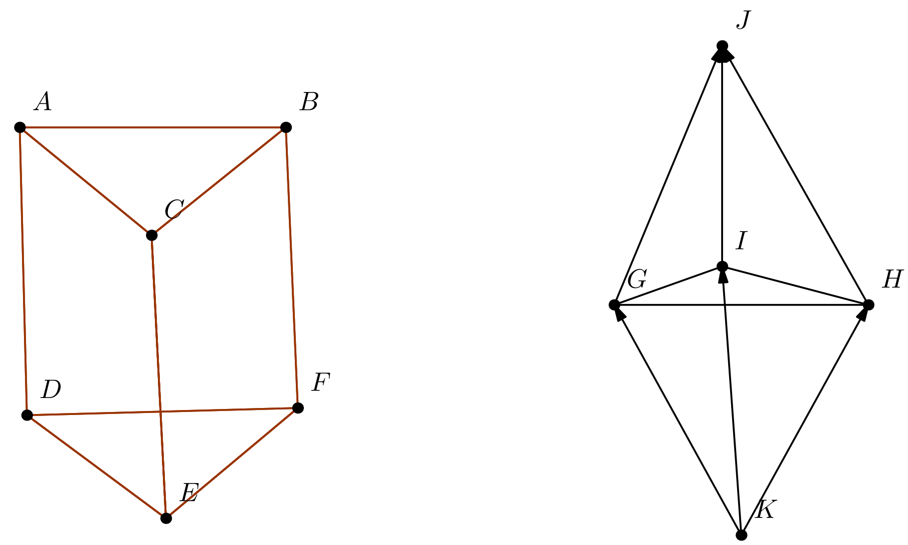

FIGURE 1

y, and three independent monotone paths from x to y, then K will be called a *generalized 3-monotone* orientation. If C is a directed cycle in K, we define a *side* of C to be a planar digraph containing C and all of the vertices and edges contained on one side of C in a plane drawing of K.

Proposition 2.1. *If C is a directed cycle in a generalized 3-monotone orientation K, then x is in one of the sides of C and y is in the other.*

Proof. The proof is by induction on the number k of faces in a given side S of C. If $k = 1$ then the proposition is vacuously true, since C is not the boundary of a face. Let e be an edge of S but not in C that has at least one vertex in C. If the tail of e is in C, we can follow a monotone path starting with e until we either arrive at y, or arrive at a vertex of C, or encounter a vertex of the monotone path for a second time. In the last two cases, we will find a directed cycle bounding a side S' that is contained in S and has fewer faces than S does. By induction, S' contains x or y. If the head of e is on C, then we can follow an antitone path and find either x or a directed cycle with a side that has strictly fewer faces than S. Because the source and the sink are unique, we must find one in each side of C. $\square$

Corollary 2.2. *Let C be a directed cycle in K. Then C contains at least one vertex of every monotone path from the source to the sink.*

Proof. This follows from Proposition 2.1 and the planarity of K. $\square$

Corollary 2.3. *Let v be a vertex of K that is neither the source nor the sink. Then there is a monotone path from v to y and an antitone path from v to x.*

Proof. If e is an edge with tail v, follow a monotone path starting with e until either the sink is reached or a directed cycle is obtained. If a directed cycle is reached this means that we have reached each of the three independent monotone $x-y$ paths. From the first intersection of our path with one of these independent monotone $x-y$ paths, we can continue on the monotone $x-y$ path to y. Similarly, we can follow an edge with head v backwards to get an antitone path from v to x. □

3. Two Lemmas of Mihalisin and Klee

In this section we follow the exposition of Mihalisin and Klee. A digraph J is *contained* in a digraph K if there exist two injections $\psi : vert(J) \to vert(K)$ and $\phi : edge(J) \to$ the set of all **monotone** paths in K satisfying the following conditions: for each edge $\overrightarrow{ab}$ of J, $\phi(\overrightarrow{ab})$ is a monotone path in K from $\psi(a)$ to $\psi(b)$; the interior of each path in image(ϕ) is disjoint from all other paths in image(ϕ). A particular choice of ψ and ϕ is called an *embedding* of J in K. Let D_4 be the unique acyclic orientation of the complete graph on four vertices.

The following lemmata, without the word "generalized," are Lemma 3.2 and Lemma 3.3 of [8]. The proofs in [8] used acyclicity only in justifying the existence of a monotone path from any vertex of a 3-monotone digraph to the sink and the existence of an antitone path to the source. Due to Corollary 2.3, the analogous property holds for generalized 3-monotone digraphs. The proofs of [8] carry over to the case of generalized 3-monotone orientations.

Lemma 3.1. *Each generalized 3-monotone digraph contains D_4. Further, the embedding may be chosen so that the source and sink of D_4 are sent to the source and sink of the digraph.*

In the following Lemma, when we say that J_{n-1} is obtained from J_n by deleting one edge, one or both of the endpoints of the deleted edge may have degree 3 in J_n and be the middle vertex of a monotone path of length 2 in J_n which is then replaced by a single edge of J_{n-1} oriented from the beginning to the end of the monotone path.

Lemma 3.2. *For each generalized* 3*-monotone digraph* K *there exists a sequence of generalized* 3*-monotone digraphs* $J_0, J_1, \ldots, J_k$ *such that* J_0 *is* D_4*,* $J_k = K$*, and each* J_{n-1} *is obtained from* J_n *by deleting one edge.*

Proof. Let J_0 be D_4, which is 3-monotone and is contained in K by Lemma 3.1. The proof of [8] shows that one can always add an edge to a 3-monotone digraph J_n contained in K to get a 3-monotone digraph digraph J_{n+1} contained in K. Their construction is also valid for generalized 3-monotone digraphs, in that J_{n+1} is contained in K, is planar and 3-connected and has 3 independent monotone paths from the source to the sink. We have to show that J_{n+1} cannot have any directed cycle contained in one of its faces. If C is a directed cycle of J_{n+1} contained in one of its faces, then C is contained in K and does not separate x and y. By replacing edges of C with monotone paths in K, this implies the existence of a directed cycle of K that does not separate x and y, contradicting Proposition 2.1. □

Corollary 3.3. *Let* K *be a generalized 3-monotone digraph. Then the restriction of* K *to any face contains a unique source and a unique sink.*

Proof. Assume the sequence $J_0, J_1, \ldots, J_k$ is as in the previous lemma. We will prove by induction that for any n, the restriction of J_n to any face contains a unique source and a unique sink. This is clearly true for $n = 0$. Suppose it is true for J_n. Because J_{n+1} is a generalized 3-monotone digraph, each of its faces contains at least one source and at least one sink. Any face of J_{n+1} with more than one source must be one of the two faces containing the edge that was added to J_n to create J_{n+1}. Call the face of J_n that was split by the new edge F and the two faces of J_{n+1} created F_1 and F_2. (See Figure 2.) If F_1 contains the source of F and F_2 contains the sink of F, then neither F_1 nor F_2 can contain a second source. If F_1 contains the source and the sink of F and F_2 contains neither, then the presence of a second source in F_1 means that F_2 has no source at all, contradicting the generalized 3-monotone property. □

4. Building the Fan

We will assume that K is a generalized 3-monotone digraph, and that we have a sequence of generalized 3-monotone digraphs $J_0, J_1, \ldots, J_k$, where J_0 is D_4, $J_k = K$, and each J_{n-1} is obtained from J_n by deleting one edge. Fix $g = (0, 0, 1)^T$. We will build a corresponding sequence $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k)$ of 3-fans so that $J_n = G_{\mathcal{F}_n,g}$ for $n = 0, 1, \ldots, k$. We may take $\mathcal{F}_0$ to be the set of cones generated by the subsets of a set of four vectors that positively span the origin and are such that g is generic with respect to the fan they generate. We have a bijection $\overline{\mu}_0$ from the set of vertices, edges and faces of D_4 to the set of 3-, 2-, and 1-dimensional cones of $\mathcal{F}_0$ that is inclusion-reversing.

For $m = 0, 1, \ldots, n$, assume that we have a fan $\mathcal{F}_m$ and a function $\overline{\mu}_m$ that is an inclusion-reversing bijection from the set of vertices, edges and faces of J_m to the set of 3-, 2- and 1-dimensional cones of $\mathcal{F}$ so that whenever an edge of J_m points from vertex v to vertex w, then $\overline{\mu}_m(w)$ and g are on the same side of the hyperplane containing $\overline{\mu}_m(\{v,w\})$ and $\overline{\mu}_m(v)$ is on the other side.

Because J_{n+1} is constructed from J_n by adding an edge which splits a face F of J_n into two faces F_1, F_2 of J_{n+1}, we will alter $\mathcal{F}_n$ by splitting the 1-dimensional cone $\overline{\mu}_n(F)$ of $\mathcal{F}_n$ into two 1-dimensional cones. Let v and w be the endpoints of the new edge of J_{n+1}. If v is a vertex of J_n, define $C(v)$ to be $\overline{\mu}_n(v)$. If v is a new vertex in the interior of an edge e of J_n, define $C(v)$ to be $\overline{\mu}_n(e)$. Define $C(w)$ similarly.

Lemma 4.1. *$C(v) \cap C(w) = \overline{\mu}_n(F)$.*

Proof. If v is a vertex of J_n, then $C(v)$ contains $\overline{\mu}_n(F)$ by the inclusion-reversing property of $\overline{\mu}_n$. If v is a new vertex in an edge e of J_n, then we will still have $C(v)$ containing $\overline{\mu}_n(F)$, because $e \subseteq F$. Thus $C(v) \cap C(w)$ contains $\overline{\mu}_n(F)$. The proof of Lemma 3.2 in [8] took great care to show that there is no edge of J_n that contains both v and w. If $C(v) \cap C(w)$ were 2- or 3-dimensional, then it would be the image under $\overline{\mu}_n$ of a vertex or edge of J_n. There is, however, no edge of J_n containing both v and w. $\square$

The following Lemma is a standard separation result. See [2], Lemma 1.13, for a close relative.

Lemma 4.2. *There is a hyperplane H in $\mathbb{R}^3$ that contains $\overline{\mu}_n(F)$ and has $C(v)\backslash\overline{\mu}_n(F)$ in one of its open halfspaces and $C(w)\backslash\overline{\mu}_n(F)$ in the complementary open half space.*

Proof. Let $C'(v)$ and $C'(w)$ be the projections of $C(v)$ and $C(w)$ onto the subspace V orthogonal to $\overline{\mu}_n(F)$. Then $C'(v)$ and $C'(w)$ are pointed cones in V. Let C be the cone generated by $C'(v) \cup -C'(w)$. We claim that this cone C is pointed. If $z \neq 0$ is such

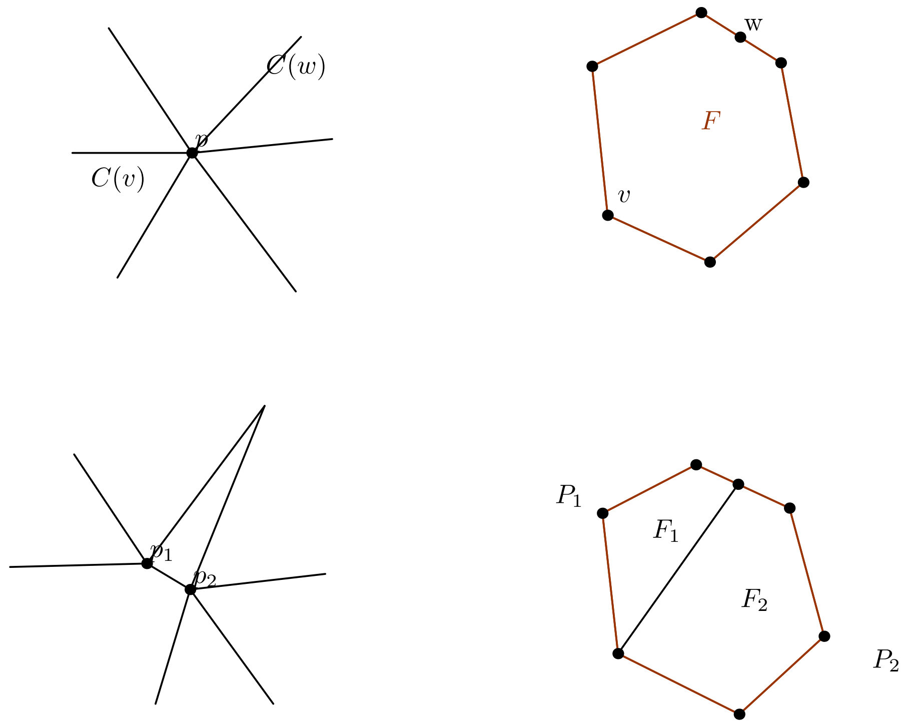

FIGURE 2

that $z \in C$ and $-z \in C$, write $z = v_1 + w_1$ and $-z = v_2 + w_2$ with $v_1, v_2 \in C'(v)$ and $w_1, w_2 \in -C'(w)$. Then $(v_1 + v_2) + (w_1 + w_2) = 0$, indicating that $v_1 + v_2 \in C'(v) \cap C'(w)$. However, $C'(v) \cap C'(w) = \{0\}$, contradicting the assumption that $z \neq 0$. Thus there is a vector u so that C is supported at $\{0\}$ by $span(u)$. It follows that $span(u)$ supports each of $C'(v)$ and $C'(w)$ at $\{0\}$ and that $C'(v)$ and $C'(w)$ are on opposite sides of $span(u)$. The hyperplane H required by the Lemma is then the span of $\{u\} \cup \overline{\mu}_n(F)$. $\square$

Let p be a nonzero vector in $\mu_n(F)$ and let $u \in V$ be as in Lemma 4.2. For $\epsilon > 0$, define $p_1 = p + \epsilon u$ and $p_2 = p - \epsilon u$. Let P_1 be the path on the boundary of F from v to w such that $p_1 \in \overline{\mu}_n(z)$ for some vertex z in the interior of P_1 and let P_2 be the other path on the boundary of F from v to w. Figure 2 shows a typical face F of J_n on the top right and the part of the fan containing the images under $\overline{\mu}_n$ of F and the vertices and edges it contains is shown on the top left. Below these are shown the corresponding objects in $\mathcal{F}_{n+1}$ and J_{n+1}. The fan $\mathcal{F}_{n+1}$ contains the following cones:

1) The one-dimensional cones generated by p_1 and p_2 and the two-dimensional cone generated by $\{p_1, p_2\}$.

2) Cones of $\mathcal{F}_n$ that do not contain p.

3) For a vertex or edge z in the interior of P_1, the cone $\overline{\mu}_n(z)$ in $\mathcal{F}_n$ is replaced by a cone with p replaced in the generating set by p_1.

4) For a vertex or edge z in the interior of P_2, the cone $\overline{\mu}_n(z)$ in $\mathcal{F}_n$ is replaced by a cone with p replaced in the generating set by p_2.

5) If v is a vertex of $J(n)$, the cone $\overline{\mu}_n(v)$ is replaced by a cone that has generator p replaced by the two generators p_1 and p_2.

6) In the case that v is not a vertex of J_n but is a new vertex in the interior of an edge e of $J(n)$, then the cone $\overline{\mu}_n(e)$ is replaced by a three-dimensional cone with p replaced in the generating set of $\overline{\mu}_n(e)$ by $\{p_1, p_2\}$.

Note that the cone generated by p is not a cone of $\mathcal{F}_{n+1}$.

Lemma 4.3. *The collection of cones $\mathcal{F}_{n+1}$ given above is a complete pointed fan, for $\epsilon > 0$ sufficiently small.*

Proof. Each cone of $\mathcal{F}_{n+1}$ that does not contain both p_1 and p_2 is a cone of $\mathcal{F}_n$ with possibly one extreme ray perturbed, so it is pointed and convex. The 3-dimensional cones that contain p_1 and p_2 are the cone generated by $\{p_1, p_2\}$ and the perturbations of $C(v)$ and $C(w)$. Because the plane spanned by $\{p_1, p_2\}$ supports $C(v)$ and $C(w)$ at p, the replacement of p in the generating set of $C(v)$ and $C(w)$ by $\{p_1, p_2\}$ will still yield a convex cone. If ϵ is small enough, these perturbations will also be pointed. $\square$

Lemma 4.4. *Suppose that the source and the sink of F are both contained in the path P_1. Then the path P_2 is a directed path, and the orientation of the new edge of J_{n+1} is determined by the requirement that there be no directed cycle in a face of J_{n+1}. If $\overline{\mu}_n(v)$ is on the same side of H as g is, then the edge containing v and w is oriented from w to v.*

Proof. That the path P_2 is a directed path is due to Corollary 3.3. Assume that the path P_2 is monotone from v to w. The images of the vertices and edges in F under $\overline{\mu}_n$ appear in the same order around $\overline{\mu}_n(F)$ as their preimages appear around F. As before, let $C'(v)$ and $C'(w)$ be the projections of $C(v)$ and $C(w)$ onto the space V orthogonal to $\overline{\mu}(p)$. Let g' be the projection of g onto V. Then $C'(v)$ and $C'(w)$ are on the same side of $span(g')$ in V. The projection of $span\{p_1, p_2\}$ onto V is $span(u)$. By construction, $span(u)$ separates $C'(v)$

and $C'(w)$. There must be a vertex q of P_2 for which the projection of $\mu_n(q)$ onto V has nonempty intersection with $span(u)$. Each of the projections of the images under $\overline{\mu}_n$ of the vertices in the directed path from q to the sink of F, including $C'(w)$, must be on the same side of u as g', and $C'(v)$ must be on the other. This agrees with the orientation of the new edge from v to w that is forced to avoid a directed cycle on a face of J_{n+1}. □

Lemma 4.5. *Suppose that the source of F is in the interior of P_1 and the sink of F is in the interior of P_2. Then we can take the hyperplane H required by Lemma 4.2 to be the span of $\{g\} \cup \overline{\mu}_n(F)$. This hyperplane can be perturbed as needed to make g generic with respect to $\mathcal{F}_{n+1}$ and to make $\overline{\mu}_{n+1}(v)$ on the same side of H as g if and only if the orientation of the edge $\{v, w\}$ in J_{n+1} is from w to v.*

Proof. Let x_F and y_F be the source and the sink of F. Then v is in the interior of one of the paths from x_F to y_F and w is in the other. The union of the projections of $\overline{\mu}_n(x_F)$ and $\overline{\mu}_n(y_F)$ onto V contains the line $span(g')$. Therefore $C(v)$ and $C(w)$ are on opposite sides of the plane spanned by g and $\overline{\mu}_n(F)$, and this plane supports them both at $\overline{\mu}_n(F)$. □

5. Counterexample in Dimension 4

Several papers, including [4], have been devoted to orientations of the graph of the d-cube. An orientation of the graph of the d-cube that is $D_{\mathcal{F},g}$ for some fan $\mathcal{F}$ and generic g is called a *PLCP-orientation.* An enumeration of the PLCP-orientations of the 4-cube was recently completed [3]. An example of an orientation of the graph of the 4-cube that is acyclic, has a unique source and sink and has 4 independent directed paths from the source to the sink but is not a PLCP-orientation was given in section 4.4 of [4]. The idea of the proof is that if the given orientation K were $D_{\mathcal{F},g}$ for some $\mathcal{F}$ and g, then for one-dimensional cones s and t of $\mathcal{F}$ corresponding to opposite facets of the 4-cube, one could obtain the fan $\mathcal{F}'$ by replacing each cone of $\mathcal{F}$ with s in its generating set by the cone with s replaced by $-s$ in the generating set, and by replacing each cone of $\mathcal{F}$ with t in its generating set by the cone with t replaced by $-t$ in the generating set. Then $D_{\mathcal{F}',g}$ would be obtained from $D_{\mathcal{F},g}$ by reversing all of the directed edges between the opposite facets of the 4-cube corresponding to s and t. However, the orientation K' obtained from K by reversing all of these edges fails to have 4 directed paths from the source to the sink. By Theorem 1.2, K' is not $D_{\mathcal{F}',g}$ for any $\mathcal{F}'$ and g.

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