Randomized Principal Pivot Algorithms for P-Matrix Linear Complementarity Problems

Walter D. Morris, jr.*

February 6, 2001

Abstract

We study orientations of the n-cube that come from simple principal pivot algorithms for the linear complementarity problem with a P-matrix. We show that these orientations properly generalize those that are obtained from linear objective functions on polytopes combinatorially equivalent to the cube. The orientations from LCP with a P-matrix may admit directed cycles. We give a sequence of problems on which the algorithm RANDOM-EDGE performs very badly.

1 Introduction and terminology

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem* (M, q) is to find nonnegative vectors w and z in $\mathbb{R}^n$ so that $w - Mz = q$ and $w^T z = 0$. A matrix $M \in \mathbb{R}^{n \times n}$ is a *P-matrix* if all of its principal minors are positive. The relevance of P-matrices to the linear complementarity problem is established by the fundamental theorem of Samelson, Thrall and Wesler:

THEOREM **1.1** *[27] A matrix M is a P-matrix if and only if the LCP (M, q) has a unique solution for all q.*

The linear complementarity problem was shown in [2] to be NP-complete. Coxson [5] showed that the problem of determining if a matrix is a P-matrix is $Co-NP$-complete. In contrast to these negative results, there is the encouraging result of Megiddo [19]:

THEOREM **1.2** *If the linear complementarity problem where M is a P-matrix is NP-hard, then $NP = Co-NP$.*

Megiddo's theorem can be regarded as a challenge to find a polynomial-time algorithm for LCPs in which the matrix M is a P-matrix. Attempts have been made to find a polynomial-time interior-point method for the LCP with a P-matrix [15], without success. The results of Kalai [12] and Matoušek et. al. [18], proving that the expected number of pivots for certain randomized pivot algorithms for linear programming is sub-exponential,

*Department of Mathematical Sciences, George Mason University, Fairfax, Virginia, 22030

prompt us to study the average-case behavior of pivot algorithms for the P-matrix LCP. The algorithms we consider are the *principal pivot algorithms* (see Chapter 4 of [24]), which are called *Bard-type algorithms* in [28] and *simple principal pivot algorithms* in [4]. These algorithms can be described, as in [28], as following directed paths on the edges of a cube.

The classic paper by Klee and Minty [14] started the study of the performance of linear programming algorithms on polytopes combinatorially equivalent to the n-cube. Much is known [8], [10], [17] about average-case behavior of simplex algorithms for linear programming when the feasible region has the combinatorial type of a cube. We show in Theorem 2.3 that the orientations of the edges of a cube given by linear functionals on polytopes combinatorially equivalent to the cube are special cases of the orientations one can obtain from principal pivot algorithms for a P-matrix LCP.

There exist orientations of a cube, coming from P-matrix LCPs, that contain directed cycles. We introduce a sequence of such problems that contain a great number of such cycles, in order to see how the presence of directed cycles affects the average number of pivots required to solve the LCP. For these problems, we find that one randomized pivot algorithm, RANDOM-EDGE, takes on the average more than $\frac{n-1}{2}!$ pivots to reach the solution on an n-cube. This is much larger than 2^n, the number of vertices of the cube. On the other hand, the algorithm RANDOM-PERMUTATION appears to take no more than $\frac{n^2+1}{2}$ pivots to solve the problem.

2 LP-cubes and P-cubes

The n-dimensional cube will be considered as a graph with vertex set $V_n = \{0,1\}^n$, and with two vertices connected by an edge if they differ in exactly one component. An orientation of the edges of the n-cube is a assignment of an ordering to each of the edges. If u is a vertex of the n-cube and $M \in R^{n\times n}$ is a matrix, then the *complementary matrix* A_u of (M^T, I) corresponding to u is the $n \times n$ matrix for which column i is I_i, the i^{th} column of the $n \times n$ identity matrix, if $u_i = 0$, and is $(M^T)_i$ if $u_i = 1$. If, in addition, $p, q \in I\!R^n$, define $\bar{A}_u$ to be the $(n+1)\times(n+1)$ matrix formed from $\mathcal{T} = \begin{pmatrix} M^T & I & 0 & p \\ -q^T & 0 & 1 & 0 \end{pmatrix}$ by letting, for $i = 1, \ldots, n$, $(\bar{A}_u)_i = \mathcal{T}_{n+i}$ if $u_i = 0$, and $(\bar{A}_u)_i = \mathcal{T}_i$ if $u_i = 1$, and setting $(\bar{A}_u)_{n+1} = \mathcal{T}_{2n+1}$. If A_u^{-1} exists, then for $i = 1, \ldots, n$ define $(\bar{N}_u)_i$ to be the column of $(\bar{A}_u)^{-1}\mathcal{T}$ that is column i if $u_i = 0$, or column $n+i$ if $u_i = 1$. For $i = 1, 2, \ldots, n$, define $(\bar{q}_u)_i$ to be the negative of the $(n+1)^{st}$ entry of $(\bar{N}_u)_i$.

DEFINITION **2.1** *An orientation of the edges of the n-cube is an LP-cube if there exists a matrix $M \in I\!R^{n\times n}$ and vectors $q, p \in I\!R^n$ so that:*

1. *for every complementary matrix A_u of (M^T, I), the system $A_u s = p$ has a unique positive solution, $(\bar{q}_u)_i$ is nonzero for $i = 1, 2, \ldots, n$, and*

2. *if vertices u and v differ in component i, then the edge $\{u, v\}$ is oriented from u to v if and only if $(\bar{q}_u)_i < 0$.*

The definition requires that there be a polytope P given by the system $M^T x \leq p$, $x \geq 0$, that is combinatorially equivalent to the n-cube, and an objective function $q^T x$ so that each edge of the cube is oriented in the direction in which $q^T x$ increases on the corresponding edge of P.

Now suppose again that $M \in \mathbb{R}^{n \times n}$, and for each vertex u of the n-cube define the matrix B_u by setting column i of B_u equal to I_i if $u_i = 0$, and $(B_u)_i = -M_i$ if $u_i = 1$.

DEFINITION **2.2** *An orientation of the edges of the n-cube is called a P-cube if there exists a P-matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ so that whenever vertices u and v differ only in component i, the edge $\{u, v\}$ is oriented from u to v if and only if $(B_u^{-1} q)_i < 0$.*

THEOREM **2.3** *Every LP-cube is a P-cube.*

Proof Suppose that we have a matrix $M \in \mathbb{R}^{n\times n}$ and vectors $p, q \in \mathbb{R}^n$ as required by Definition 2.1. Cramer's rule and the positivity of the solutions to $A_u s = p$ imply that the determinants of the matrices A_u all have the same sign. Thus M^T and hence M is a P-matrix.

Consider the matrix $\mathcal{S} = \begin{pmatrix} I & -M & q & 0 \end{pmatrix}$. Note that the rows of $\mathcal{S}$ are in the null space of $\mathcal{T}$. This implies that the entries of row i of $B_u^{-1}\mathcal{S}$ that are not in columns corresponding to B_u are the negatives of the entries of column $(\bar{N}_u)_i$ of $\mathcal{T}$. In particular, it will be true that the entry in row i, column $2n+1$ is $(\bar{q}_u)_i$. ∎

The first study of the combinatorial properties of P-cubes is [28]. Suppose that Q is an n-dimensional P-cube and suppose that $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are as in Definition 2.2. We then see that a vertex u of Q is a sink in Q (has outdegree 0) if and only if the there is a solution (w, z) to the LCP $w - Mz = q, w, z \geq 0, w^T z = 0$ with $w_i > 0$ only if $u_i = 0$ and $z_i > 0$ only if $u_i = 1$. The theorem of Samelson, Thrall, and Wesler shows that there may only be one sink in Q. Furthermore, as [28] shows, every face of a P-cube is a P-cube. Therefore every face of a P-cube has a unique sink. Another important fact proved in [28] is that if v is the sink of a P-cube and u is a vertex that differs from v in k components, then there exists a directed path of length k from u to v.

The relationship between the LCP $w - Mz = q, w, z \geq 0, w^T z = 0$ and the linear program $max\ q^T x : M^T x \leq p, x \geq 0$ was first noted by Mangasarian [16]. Mangasarian's goal was to find conditions on M under which one could find a vector p so that a solution to the linear program determined a solution to the linear complementarity proplem. The special case in which such a p exists, when the complementary matrices of (M^T, I) correspond to nondegenerate feasible bases of the linear program, has been further investigated by several authors [13], [25],[21], [20]. In this case, the matrix M is called a *Hidden Minkowski* matrix and the vector p is called an *n-step vector* for M^T.

The orientations of the 3-cube for which every face has a unique sink were enumerated by [28]. There is one such orientation that contains a cycle (see Figure 2.1), and [28] showed that it was a P-cube with $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

The enumeration of [28] also showed that there were two acyclic orientations of the 3-cube in which every face had a unique sink but which were not P-cubes. (See Figure 2.2.)

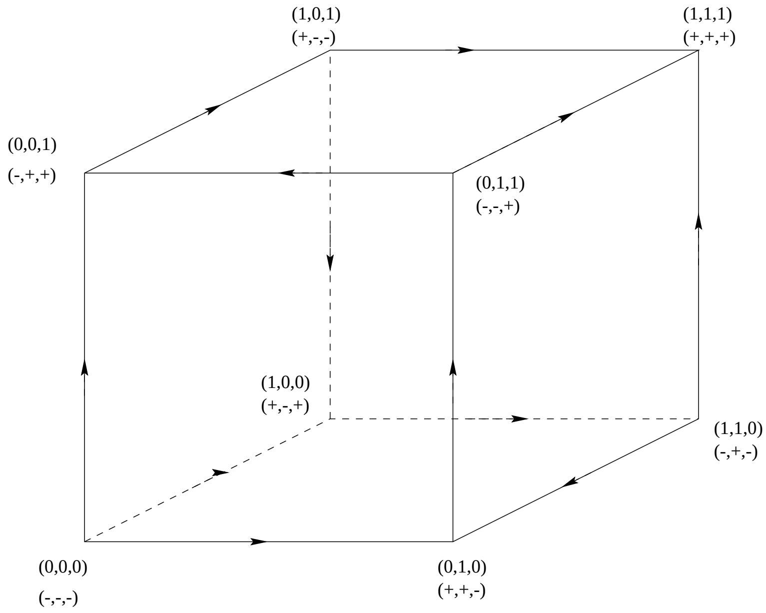

Figure 1: A P-cube with a directed cycle

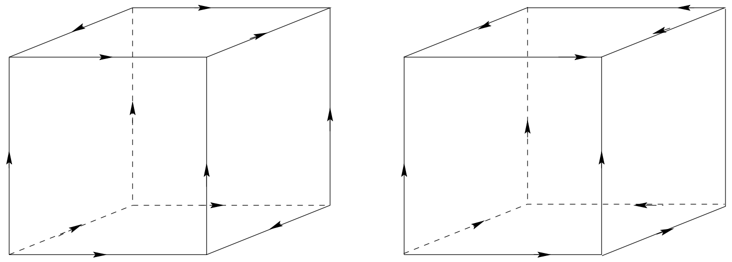

Figure 2: Two cubes that are not P-cubes

Gärtner and Kaibel [9] independently enumerated the acyclic orientations of the 3-cube in which every face had a unique sink. They discovered that all of the orientations were LP-cubes except those of Figure 2.2. To prove that these are not LP-cubes, [8] points out that neither of these cubes contains a set of three disjoint directed paths connecting the source and the sink. By a theorem of [11], one can conclude that these are not LP-cubes. We are left with two natural open problems for P-cubes. First, the problem of finding an acyclic P-cube that is not an LP-cube, and then the problem of determining if every n-dimensional P-cube must contain a set of n disjoint directed paths connecting the source and sink.

Matoušek [17] discovered a family of acyclic orientations of cubes, in which each face had a unique source and sink, so that for a randomly chosen n-cube in his family, the expected number of pivots used by the algorithm RANDOM-FACET is bounded *below* by $\frac{1}{n}e^{\sqrt{2n}}$. Gärtner, however, showed that if one removes from Matoušek's family those cubes that contain one of the cubes of Figure 2.2, the expected number of pivots becomes quadratic. That these two cubes are not P-cubes leaves the possibility that RANDOM-FACET is a randomized polynomial pivot algorithm for P-cubes open.

To conclude this section, we need to point out that *oriented matroid programs* for which the feasible region is a cube yield another way to generalize LP-cubes so that cycles are allowed. Compare, for example, Figure 2.1 to Figure 10.4.7 of [1]. It seems likely that there are P-cubes that do not arise from oriented matroid programs, as well as cube orientations arising from oriented matroid programs that are not P-cubes. No examples of either possibility are known.

3 Randomized Principal Pivot Algorithms

Let Q be an n-dimensional P-cube. A *principal pivot algorithm* takes as input a vertex u of Q and produces a sequence $(u = u^0, u^1, \dots)$ of vertices of Q, where for $j = 1, 2, \dots$, vertices u^{j-1} and u^j are adjacent and the edge containing them is oriented from u^{j-1} to u^j. If the sequence produced by the algorithm is finite, we require that it terminates at the unique sink of Q. The *length* of a finite sequence $(u = u^0, u^1, \dots, u^k)$ produced by a principal pivot algorithm is k. The sequence produced by a principal pivot algorithm on an acyclic Q must be finite, with $k \leq 2^n - 1$.

A vertex of Q may have many edges directed away from it. There are several different specialized principal pivot algorithms that have been proposed [24]. We are particularly interested in *combinatorial* algorithms, which use only the information in Q.

As we saw in section 2, a P-cube may contain a directed cycle. A principal pivot algorithm running on such a P-cube may produce an infinite sequence. The following principal pivot algorithm of Murty [22] produces a finite sequence even if the P-cube contains a directed cycle.

ALGORITHM **3.1** *For $j = 0, 1, 2, \dots$, vertex u^{j+1} is obtained from u^j by changing component i_j, where $i_j = min\{i :$ the edge from u^j to the vertex obtained by changing component i is directed away from $u^j\}$.*

Even though Murty's algorithm always produces a finite sequence terminating in the sink of Q, the length of such a sequence may be as large as $2^n - 1$. Murty [23] himself gave upper triangular examples of such Q. For $n = 1, 2, \dots$, let

$$M^n = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 1 & 2 & \cdots & 2 & 2 & 2 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, q^n = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \end{pmatrix}.$$

Fukuda and Namiki [7] proposed to apply a randomized algorithm to Murty's sequence of problems. We will call their algorithm RANDOM-PERMUTATION.

ALGORITHM **3.2** *Pick a permutation* π *of* $\{1, 2, \ldots, n\}$ *at random. For* $j = 0, 1, 2, \ldots,$ u^{j+1} *is obtained from* u^j *by changing component* i_j*, where* $\pi(i_j) = min\{\pi(i)$: *the edge from* u^j *to the vertex obtained by changing component* i *is directed away from* $u^j\}$.

The algorithm RANDOM-PERMUTATION is Murty's algorithm, starting with a random permutation of the coordinates of the cube. This algorithm is closely related to Kalai's ALGORITHM II (see [12]). The arguments of [8] apply and show that if Q is acyclic, then the expected length of the sequence created by RANDOM-PERMUTATION is bounded above by $e^{2\sqrt{n}-1}$.

Fukuda and Namiki [7] showed that the expected length of a sequence produced by RANDOM-PERMUTATION on the n-cube of Murty's sequence is n. A comparison of their combinatorial description of Murty's n-cubes with the description of the Klee-Minty cubes given in [10] shows that these n-cubes are the same. See [3] for a combinatorial analysis of Murty's examples and related linear complementarity problems.

The algorithm RANDOM-FACET is related to Kalai's [12] ALGORITHM II and to the algorithm of Matoušek, Sharir, and Welzl [18]. Some subtle differences among these algorithms are discussed in [6]. We repeat the description from [10]: If there is only one edge directed away from u, take it. Otherwise, consider the restricted problem on a randomly chosen facet containing u. Solve the restricted problem by a recursive call to RANDOM-FACET. Then repeat with the vertex obtained from the recursive call. It is shown in [8] that the expected length of the sequence created by RANDOM-FACET for an acyclic P-cube is bounded above by $e^{2\sqrt{n}-1}$. The algorithm RANDOM-FACET applied to the Klee-Minty cubes yields a sequence of expected length n (see [10] for more).

The last algorithm we consider is RANDOM-EDGE. Here we give each of the edges leaving u^j equal probability in choosing the one on which to leave u^j. No subexponential upper bound on the expected length of the sequence returned by RANDOM-EDGE applied to acyclic cubes is known, but the algorithm seems to perform well on the Klee-Minty cube [10].

Each of the algorithms RANDOM-EDGE and RANDOM-FACET may return an infinite sequence when applied to the example of Fig. 2.1. The expected length of the sequence will, however, be finite. For the algorithm RANDOM-EDGE, the state of the algorithm when vertex u^j has been determined is completely described by the vertex u^j. For a vertex v, we define $E(v)$ to be the expected length of a path produced by the algorithm starting at v. If v is not the sink of the cube, we then consider the equation $E(v) = 1 + \sum(1/k)E(w)$, where the sum is over vertices w for which there is an edge directed from v to w, and k is the number of edges leaving v. If v is the sink of the cube, we write $E(v) = 0$. This gives us a system of 2^n equations in 2^n variables to solve. The computation of the expected length of a sequence returned by RANDOM-FACET is more complicated, because the state of the algorithm after vertex u^j has been determined depends not only on u^j, but on the nested sequence of subproblems that is being considered at the moment when u^j is chosen. For every possible state of the algorithm, one can theoretically write down an equation such as the one considered above, and solve the resulting enormous system of equations. One can show that in both cases, the systems have solutions and thus the expected values are finite. The situation for RANDOM-PERMUTATION is simpler. One averages the (finite!) lengths of the sequences generated by the algorithms for the $n!$ choices of π.

At this point the reader might want to verify that for the cube of Figure 2.1, the expected length of a sequence returned by the algorithms RANDOM-PERMUTATION and RANDOM-EDGE, starting at $(0,0,0)$, is 5, while the expected length of a sequence returned by RANDOM-FACET is $5\frac{1}{2}$.

4 A sequence of highly cyclic examples

For all of this section, for $n = 1, 2, \dots,$ let

$$M^n = \begin{pmatrix} 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, q^n = \begin{pmatrix} -1 \\ -1 \\ \\ \vdots \\ \\ -1 \\ -1 \end{pmatrix}.$$

The matrix M^n is an $n \times n$ circulant matrix. There are two nonzero terms in the expansion of the determinant of M^n. M^n will be a P-matrix if and only if n is odd. For $n = 3$, we have the example of Figure 2.1. Because of the circulant structure of M^n, we will consider all subscripts in this section to be taken modulo n.

Let u be a vertex of the n-cube, and let B_u be a complementary submatrix of the matrix $(I, -M^n)$. Recall that this means that B_u is an $n \times n$ matrix, and column i of B_u is I_i if $u_i = 0$ and $-M^n_i$ if $u_i = 1$. We would like to be able to describe the sign pattern of the solution $B_u s = q^n$ in terms of the vertex u. Therefore, we will define the n-vector $\sigma(u)$ by setting $\sigma(u)_i = +$ if $(B_u^{-1} q^n)_i > 0$ and $\sigma(u)_i = -$ if $(B_u^{-1} q^n)_i < 0$. In Figure 2.1, we have displayed the vector $\sigma(u)$ beneath u for each vertex of the 3-cube.

We first note that the solution to $-M^n s = q^n$ is $(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$. This means that for $u = (1,1,\dots,1)$, we have $\sigma(u) = (+,+,\dots,+)$. Next, suppose that C is a $k \times k$ matrix of the form

$$C = \begin{pmatrix} 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -2 & & 0 & 0 & 0 \\ 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & & -1 & -2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix}.$$

The matrices C and $-M^k$ differ only in positions $(1,1)$ and (k,1). The solution s to $Cs = q^k$ has $s_i = \frac{1-(-2)^{k-i+1}}{3}$ for $i = 2, 3, \dots, k$, and $s_1 = -\frac{1-(-2)^k}{3}$. This means that the last component of s is positive, and the components $2, 3, \dots, k$ of s alternate in sign, but the first and second components of s agree in sign. We call a *block* of u a sequence $(u_{i+1}, u_{i+2}, \dots, u_{i+k})$, where $k \in \{1, 2, \dots, n\}$, $u_{i+1} = 0$, $u_{i+2} = u_{i+3} = \cdots = u_{i+k} = 1$, $u_{i+k+1} = 0$. If $u \neq (1,1,\dots,1)$, then every component of u is contained in a unique block.

LEMMA **4.1** *Let u be a vertex of the n-cube, and suppose that for some $i \in \{1, 2, \ldots, n\}$ we have $u_{i+1} = 0, u_{i+2} = u_{i+3} = \cdots = u_{i+k} = 1, u_{i+k+1} = 0$. Then $\sigma(u)_{i+1} = \sigma(u)_{i+2}, \sigma(u)_{i+k} = +$, and for $l = 2, 3, \ldots, k-1$, we have $\sigma(u)_{i+l} \neq \sigma(u)_{i+l+1}$.*

Proof The system $B_u s = q^n$ is decomposable, so that the components $i+1, i+2, \ldots, i+k$ of s can be computed independently of the other components of s, by solving a system of the form $Cs = q^k$. ■

For a vertex u in the n-cube, define $\chi(u)$ to be the number of components of u that are equal to 1 plus the number of components of $\sigma(u)$ that are equal to $+$. Thus $\chi((0, 0, \ldots, 0)) = 0$, while $\chi((1, 1, \ldots, 1)) = 2n$.

LEMMA **4.2** *If u and v are adjacent vertices of the cube, differing in component i, and $\sigma(u)_i = -$, then $\chi(v) - \chi(u) = 0$ or 3, except when $v = (1, 1, \ldots, 1)$, in which case $\chi(v) - \chi(u) = \frac{n+3}{2}$.*

Proof Suppose $v \neq (1, 1, \ldots, 1)$. If $u_i = 1$ and $\sigma(u)_i = -$, then $v_i = 0$ and $\sigma(v)_i = 1$. On all other components, u agrees with v and $\sigma(u)$ agrees with $\sigma(v)$. Therefore $\chi(u) = \chi(v)$. If $u_i = 0$, we may assume that there exists $k \in \{1, 2, \ldots, n\}$ so that $u_k = 0$, $u_{k+1} = u_{k+2} = \cdots = u_{i-1} = 1$. (Recall that all subscripts are modulo n). Then $\sigma(u)$ and $\sigma(v)$ will disagree on exactly those coordinates that are in $\{k, k+1, \ldots, i\}$. If $\sigma(u)_k = \sigma(u)_{k+1} = 0$, then $\chi(v) - \chi(u) = 3$, while if $\sigma(u)_k = \sigma(u)_{k+1} = 1$, then $\chi(v) - \chi(u) = 0$. If $v = (1, 1, \ldots, 1)$, then we may assume that $u = (0, 1, 1, \ldots, 1)$. In that case, $\chi(u) = 3\frac{n-1}{2}$ and $\chi(v) = 2n$, so $\chi(v) - \chi(u) = \frac{n+3}{2}$. ■

It follows from Lemma 4.2 that if $(u^0, u^1, u^2, \ldots)$ is a sequence produced by a principal pivot algorithm for the LCP (M^n, q^n), then $\chi(u^0) \leq \chi(u^1) \leq \chi(u^2) \leq \cdots$. The function χ serves as an objective function, but one with very large level sets. There are $\frac{n+3}{2}$ level sets for χ. We will concentrate on the level set $\psi(n)$, consisting of vertices u of the n-cube for which $\chi(u) = 3\frac{n-1}{2}$.

LEMMA **4.3** *If $u \in \psi(n)$, then there exists exactly one $m \in \{1, 2, \ldots, n\}$ for which $u_m = 0$ and $\sigma(u)_m = -$.*

Proof A block with k even contributes $3\frac{k}{2}$ to $\chi(u)$, while a block with k odd contributes $3\frac{k-1}{2}$ to $\chi(u)$. In order for $\chi(u)$ to be $3\frac{n-1}{2}$, it follows that there must be exactly one block with k odd. For such a block $(u_{i+1}, u_{i+2}, \ldots, u_{i+k})$ with k odd, $\sigma(u)_{i+1} = -$, whereas $\sigma(u)_{i+1}$ would be $+$ if k were even. ■

PROPOSITION **4.4** *The expected length of a sequence returned by the algorithm RANDOM-EDGE for the LCP (M^n, q^n), starting at the vertex $(0, 1, 1, \ldots, 1)$ adjacent to the solution, is bounded below by $\frac{n-1}{2}!$.*

Proof For $u \in \psi(n)$, define $\phi(u) = |\{i : \sigma(u)_i = -\}|$. $m = 1, 2, \ldots, \frac{n+1}{2}$, define $E(m)$ to be the expected length of a sequence returned by RANDOM-EDGE, starting from a vertex u of $\psi(n)$ with $\phi(u) = m$. From Lemma 4.3, we see that for $u \in \psi(n)$, with $phi(u) = m$, one of the edges leaving u goes to a vertex v with one more 1 component than u, so $\phi(v) = m + 1$, unless $v = (1, 1, \ldots, 1)$. The other $m - 1$ edges leaving u go to vertices w with $\phi(w) = m-1$. We obtain the equations $E(1) = 1 + E(2)$, $E(m) = 1 + \frac{1}{m}E(m+1) + \frac{m-1}{m}E(m-1)$ for $m = 2, 3, \ldots, \frac{n-1}{2}$, and $E(\frac{n+1}{2}) = 1 + \frac{n-1}{n+1}E(\frac{n-1}{2})$. We have the tridiagonal system $R^n x^n = e^n$, where

$$R^n = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & -\frac{2}{n-3} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{n-3}{n-1} & 1 & -\frac{2}{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{n-1}{n+1} & 1 \end{pmatrix}, e^n = \begin{pmatrix} 1 \\ 1 \\ \\ \vdots \\ \\ 1 \\ 1 \end{pmatrix}.$$

Note that the system for n is obtained from that for $n-2$ by appending a row and column. One can show by induction that $x^n_{\frac{n+1}{2}} = \frac{n+1}{2} + \frac{n-1}{2}x^{n-2}_{\frac{n-1}{2}}$. This implies that $x^n_{\frac{n+1}{2}} > \frac{n-1}{2}!$. ■

The conclusion one can draw from this is that the algorithm RANDOM-EDGE performs very badly on the LCP (M^n, q^n). Because $\frac{n-1}{2}!$ is asymptotically much larger than 2^n, we see that RANDOM-EDGE can be expected to travel through many cycles before reaching the solution.

For $n = 5$ with starting vertex $(0,0,0,0,0)$, the expected lengths of paths returned by RANDOM-EDGE, RANDOM-FACET, and RANDOM-PERMUTATION are, respectively, $15.2, 11.9$, and 10.5. We have had no success in analyzing the algorithm RANDOM-FACET for the LCP (M^n, q^n), if $n > 5$.

Our empirical observations for the algorithm RANDOM-PERMUTATION for the LCP (M^n, q^n) have, on the other hand, been quite encouraging. One can find, for any n, permutations of the set $\{1, 2, \ldots, n\}$ which force Murty's algorithm to take either $\frac{n^2+1}{2}$ or $2n-1$ steps to reach the solution, starting from the origin. For $n \leq 9$ we have verified that these are the maximum and minimum lengths of paths generated by RANDOM-PERMUTATION. It seems reasonable to conjecture that the same situation prevails for $n > 9$.

The conclusion that we reach is that the algorithm RANDOM-PERMUTATION is an algorithm worth further study. RANDOM-EDGE, on the other hand, seems only to be useful for the acyclic case.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, Cambridge University Press, Cambridge, 1993.

[2] S.-J. Chung, *A note on the complexity of LCP: The LCP is NP-complete*, Technical Report 79-2 (1979), Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor.

[3] R. W. Cottle, *Observations on a class of nasty linear complementarity problems*, *Discrete Applied Mathematics*, **2** (1980) 89 - 111.

[4] R. W. Cottle, J.-S. Pang and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

[5] G. Coxson, *The P-matrix problem is Co-NP complete*, *Mathematical Programming*, **64** (1994) 173 - 178.

[6] L. Finschi, K. Fukuda and H.-J. Lüthi, *Toward a unified framework for randomized pivoting algorithms in linear programming*, in P. Kall and H.-J. Lüthi, eds., *Operations Research Proceedings 1998* (1999), 113 - 122.

[7] K. Fukuda and M. Namiki, *On extremal behaviors of Murty's least index method*, *Mathematical Programming* **64** (1994) 365 - 370.

[8] B. Gärtner, *Combinatorial linear programming: geometry can help* Lecture Notes in *Computer Science* **1518**, (1998) Springer-Verlag, 82 - 96.

[9] B. Gärtner and V. Kaibel, *Abstract objective function graphs on the 3-cube - a classification by realizability* Tech. Report 296, (1998) Dept. of Computer Science, ETH Zürich.

[10] B. Gärtner, M. Henk and G. Ziegler, *Randomized simplex algorithms on Klee-Minty cubes*, *Combinatorica* **18** (1998)

[11] F. Holt and V. Klee, *A proof of the strict monotone 4-step conjecture*, In J. Chazelle, J. B. Goodman and R. Pollack, eds., *Advances in Discrete and Computational Geometry*, Contemporary Mathematics, American Mathematical Society (1998).

[12] G. Kalai, *Linear programming, the simplex method and simple polytopes*, *Mathematical Programming*, **79** Elsevier North-Holland (1997), 217 - 233.

[13] L. M. Kelly, K. G. Murty and L. T. Watson, *CP-rays in simplicial cones*, *Mathematical Programming* **48** (1990) 387-414.

[14] V. Klee and G.J. Minty, *How good is the simplex algorithm?*, in *Inequalities III*, O. Shisha, ed., Academic Press, New York (1972) 159 - 175.

[15] M. Kojima, N. Megiddo and Y. Ye, *An interior point potential reduction method for the linear complementarity problem*, *Mathematical Programming* **54** (1992), 267 - 279.

[16] O. L. Mangasarian, *Linear complementarity problems solvable by a single linear program*, *Mathematical Programming* **10** (1976) 263 - 270.

[17] J. Matoušek, *Lower Bounds for a subexponential optimization algorithm*, *Random Structures and Algorithms* **5** (1994) 591 - 607.

[18] J. Matoušek, M. Sharir and E. Welzl, *A subexponential bound for linear programming*, *Proceedings of the eighth annual symposium on Computational Geometry*, ACM, Berlin 6 (1992), 1 - 8.

[19] N. Megiddo, *A note on the complexity of P-matrix LCP and computing an equilibrium*, Research Report (1988), IBM Almaden Research Center, San Jose.

[20] W. D. Morris, Jr., *Extended P-pairs*, *Linear Algebra and its Applications*, **257** (1997) 211 - 223.

[21] W. D. Morris, Jr. and J. Lawrence, *Geometric properties of hidden Minkowski matrices*, *SIAM Journal on Matrix Analysis and its Applications* **10** (1988) 229 - 232.

[22] K. G. Murty, *Note on a Bard-type scheme for solving the complementarity problem*, *Opsearch* **11** (1974) 123 - 130.

[23] K. G. Murty, *Computational complexity of complementary pivot methods*, Mathematical Programming Study **7** (1978) 61 - 73.

[24] K. G. Murty, *Linear Programming, Linear and Nonlinear Programming*, (1988) Heldermann Verlag, Berlin.

[25] J.-S. Pang, *On discovering hidden Z-matrices*, in Constructive Approaches to Mathematical Models, C. V. Coffman and G. J. Fix, eds., Academic Press, New York (1979), 231 - 241.

[26] J.-S. Pang and R. Chandrasekaran, *Linear complementarity problems solvable by a polynomially bounded pivoting algorithm*, Mathematical Programming Study **25** (1985), 13 - 27.

[27] H. Samelson, R. M. Thrall and O. Wesler, *A partition theorem for Euclidean n-space*, *Proceedings of the American Mathematical Society*, **9**, 805 - 807.

[28] A. Stickney and L. T. Watson, *Digraph models of Bard-type algorithms for the linear complementarity problem*, Mathematics of Operations Research **3**, 322-333.