Good hidden P-matrix sandwiches

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Abstract

We introduce two new classes of matrices that contain the class of P-matrices and can be recognized in polynomial time. These classes are related to the known classes of Hidden Minkowski and hidden positive row diagonally dominant matrices, which are subclasses of the P-matrix class. We show that for 3×3 matrices, the class of hidden prdd matrices coincides with the class of P-matrices. Alternative theorems for these hidden matrix classes are given and it is shown how one can use them to efficiently check membership in the classes.

1 Introduction

An $n \times n$ real square matrix is called a *P-matrix* if its all principal minors are positive. The problem of deciding if a given matrix A is a P-matrix is known to be *co*-NP-complete [1]; thus there seems to exist no good (in the sense of being checkable in polynomial time) characterization of P-matrices. Despite the intractability, detecting P-matrices has been an object of study [2]. One way to address this hard problem is to study matrix classes

Key words. P-matrices, hidden Minkowski matrices, row diagonally dominant, alternative theorem, linear complementarity problems

that have good characterizations and that either contain or are contained in the class of P-matrices.

A major motivation for studying P-matrices is that the complexity of solving a linear complementarity problem (LCP) (see [3, 4]) when the matrix A is a P-matrix is unresolved. An LCP defined by a square matrix A and a vector q, is to find nonnegative vectors x and y so that $y = Ax + q$ and $x^T y = 0$. Megiddo [5] showed that if the problem of solving an LCP with a P matrix A were NP-hard, then it would be true that the complexity classes NP and *co*-NP would be equal. This has led to some optimism for finding a polynomial time algorithm.

A subclass of the class of P-matrices that has attracted much attention in linear complementarity theory is that of the hidden Minkowski matrices. Polynomial time algorithms for LCPs are known [6, 7] if the matrix A is hidden Minkowski or if its transpose is hidden Minkowski, but the algorithms are different. (See also [8], [9].) In 1995, Johnson and Tsatsomeros [10] introduced a very interesting new subclass of the class P, which we call hidden positive row diagonally dominant or hidden *prdd*. This matrix class properly contains the class of hidden Minkowski matrices, for $n \geq 3$. In order to apply what is known about solving LCPs with hidden Minkowski matrices to LCPs with hidden prdd matrices, it is important to understand the relationships between these matrix classes. This has recently become more relevant due to the work of Svensson and Vorobyov [11], who showed that if one could solve, in polynomial time, generalized LCPs with all representative submatrices positive row diagonally dominant, then one could also solve simple stochastic games in polynomial time. (See the end of Section 5 for details.)

Additional motivation for studying the hidden prdd class comes from the conjecture, reported in [2], that every P-matrix is hidden prdd. While a 4×4 counterexample was randomly generated in [12], many questions were left unanswered. We give more counterexamples, generated more systematically, in section 7. We also prove, in section 5, that every 3×3 P-matrix is hidden prdd. Section 7 shows that the hidden prdd class is not closed under taking transposes.

In Section 2, we derive theorems of the alternative for many of the matrix classes used in the paper, similar to the alternative theorem for the hidden prdd class proved in [12]. These theorems allow us to give short certificates of non-membership in the relevant classes.

We use the concept of *hidden* matrix classes as a unifying framework. If $\mathcal{C}$ is a class of square matrices, we will say that a matrix A is hidden $\mathcal{C}$ if there exist matrices C and B in $\mathcal{C}$ so that $AC = B$. The *prdd* and *prdd*-∞ matrices are those for which, in each row, the diagonal entry is strictly larger than the 1-norm and the ∞-norm of the row with the diagonal element removed. We show in section 3 that every P-matrix is hidden prdd-∞ and give a new proof to show that every hidden prdd matrix is a P-matrix.

Section 4 investigates hidden matrix classes between hidden prdd and hidden prdd-∞, corresponding to different norms, attempting to approximate the class P more closely. We investigate k-prdd matrices, defined using polyhedral norms, and prdd-p norms, which are defined by the usual p-norms.

A slight problem with our use of the word hidden is that it is not always true that a matrix A is hidden Minkowski (as the term is used in the literature) if there are Minkowski matrices C and B so that $AC = B$. We show in Section 5, however, that there is a class, which we call Z^o-, so that hidden Minkowski matrices are exactly the hidden Z^o--

matrices, with the last *hidden* as in our definition. We introduce a new class, called Z^o+, which has a definition remarkably similar in form to that of Z^o-, and show that all P-matrices are hidden Z^o+. We also show that all 3×3 hidden Z^o+-matrices are hidden prdd, and hence these two classes coincide with the class of P-matrices, for the 3×3 case.

Membership in all of the classes hidden prdd, hidden prdd-∞, hidden Z^o- and hidden Z^o+ can be checked in polynomial time.

Section 6 puts theorems of Horn and Johnson and of Metzler into our context. Section 7 gives examples to show, using our alternative theorems, that some of the inclusions we prove in the paper are strict.

Throughout this paper we use upper-case letters to represent matrices and sets of indices and lower-case letters for vectors and scalars. The capital N will always denote the set $\{1, 2, \ldots, n\}$. Each entry of a matrix A indexed by i and j is addressed by A_{ij}. For a matrix A, letters $A_{i\cdot}$ and $A_{\cdot j}$ represent the i-th row vector and the j-th column vector. The letter e denotes the vector with all entries equal to 1 and the letter e_j denotes the unit vector with component j equal to 1. Upper case I is always the identity matrix. For $i, j \in N$, define E_{ij} to be the $n \times n$ matrix with element (i, j) equal to 1 and all other elements equal to 0. For a subset $G \subseteq N$, define $e^G \in \mathbb{R}^n$ to be the vector with -1 in the coordinates indexed by G and 1 in the remaining coordinates.

2 Alternative theorems for hidden matrices

Duality plays a central role in our investigation. Among the many equivalent theorems of the alternative, the most adaptable for us seems to be the following.

Theorem 1 (Rockafellar, Theorem 11.2 [13]) *Let C be a nonempty relatively open convex set in $\mathbb{R}^n$, and let M be a nonempty affine set in $\mathbb{R}^n$ not meeting C. Then there exists a hyperplane H containing M, such that one of the open half-spaces associated with H contains C.*

Let us define the scalar product of matrices (R, S) and (B, C) in $\mathbb{R}^{n \times 2n}$ by $(R, S) \cdot (B, C) := tr((R, S)^T (B, C))$. This is the usual inner product if one considers the matrices to be vectors of length $2n^2$. For a given matrix $A \in \mathbb{R}^{n \times n}$, the set M_A of matrices (C, B), such that C, B are in $\mathbb{R}^{n \times n}$ and $AC = B$, is a subspace of $\mathbb{R}^{n \times 2n}$ spanned by the matrices (E_{ij}, AE_{ij}). Note that AE_{ij} is the matrix with column i of A in its j^{th} column and zeros everywhere else. It follows that $(R, S) \cdot (E_{ij}, AE_{ij}) = R_{ij} + (A^T S)_{ij}$. The set of matrices (R, S) that satisfy $R + A^T S = 0$ is therefore the orthogonal complement $M_A^\perp$ of M_A. The set of matrices (R, S) that satisfy $(R, S) \cdot (C, B) \geq 0$ for all C, B in a convex cone K is called the dual cone of $K \times K$ and is denoted $(K \times K)^*$.

Theorem 2 *Let K be a nonempty, open convex cone in $\mathbb{R}^{n \times n}$. Let $A \in \mathbb{R}^{n \times n}$. Exactly one of the following holds:*

(a) There exists (C, B) in $(K \times K) \cap M_A$.

(b) There is a nonzero (R, S) in $M_A^\perp \cap (K \times K)^*$.

Proof. Note that because K is a nonempty, open and convex cone, it follows that $K \times K$ is a nonempty, open, convex cone. If M_A misses $K \times K$ then Theorem 1 states that there is a hyperplane H (corresponding to (R, S)) containing M_A so that one of the open half-spaces associated with H contains $K \times K$. Because $K \times K$ is an open set in $\mathbb{R}^{n\times 2n}$, this is equivalent to saying that $K \times K$ is in one of the closed half-spaces associated with H. Conversely, if there is a nonzero (R, S) in $M_A^{\perp}$ so that $(R, S) \cdot (B, C) \geq 0$ for all (B, C) in $K \times K$, then $(R, S) \cdot (B, C) > 0$ for all (B, C) in $K \times K$ because $K \times K$ is open. Therefore, $M_A \cap (K \times K) = \emptyset$. ∎

Definition 1 (prdd, prdd-∞, Z^o- and Z^o+ cones and matrices)

1. $K_{prdd} := \{C \in \mathbb{R}^{n\times n} | C_{ii} > \sum_{j\neq i} |C_{ij}| \text{ for all } i\}$,
2. $K_{prdd\text{-}\infty} := \{C \in \mathbb{R}^{n\times n} | C_{ii} > |C_{ij}| \text{ for all } i \neq j\}$,
3. $K_{Z^o-} := \{C \in \mathbb{R}^{n\times n} | C_{ij} < 0 \text{ whenever } i \neq j, \text{ and } Ce > 0\}$,
4. $K_{Z^o+} := \{C \in \mathbb{R}^{n\times n} | C_{ij} < 0 \text{ whenever } i \neq j, C_{ii} > 0 \text{ for all } i \text{ and } Ce < 0\}$.

A matrix C in K_{prdd} ($K_{prdd\text{-}\infty}$, K_{Z^o-}, K_{Z^o+}) is simply called a prdd (prdd-∞, Z^o-, Z^o+) matrix. We note that prdd-p is the abbreviation of positive row diagonally dominant with respect to the p-norm, and we omit the trailing p when $p = 1$. All matrices in the classes Z^o- and Z^o+ have positive diagonal elements and negative off-diagonal elements. We will later define (Definition 5) the class of matrices with this sign pattern to be the class Z^o.

Definition 2 (hidden prdd, prdd-∞, Z^o- and Z^o+ matrices) *A matrix $A \in \mathbb{R}^{n\times n}$ is called hidden prdd (prdd-∞, Z^o-, Z^o+) if there are matrices C and B in K_{prdd} ($K_{prdd\text{-}\infty}$, K_{Z^o-}, K_{Z^o+}) so that $AC = B$.*

Each of the cones listed in Definition 1 is an open polyhedral cone in the space of real $n \times n$ matrices. It is well known and easily proved that $(K \times K)^* = K^* \times K^*$, i.e. a matrix (R, S) is in the dual cone of $K \times K$ if and only if $(R, S) \cdot (0, P) \geq 0$ and $(R, S) \cdot (P, 0) \geq 0$ for every generator P of K. In order to apply Theorem 2 to the cones of Definition 1 and compute their dual cones, we characterize the cones in terms of their extreme rays.

Proposition 1 *Let $C \in \mathbb{R}^{n\times n}$.*

1. *$C \in K_{prdd}$ if and only if there exist $y_{ij} > 0$ and $z_{ij} > 0$ for all $i, j \in N, i \neq j$ and such that $C = \sum_{i\neq j}(y_{ij}(E_{ii} - E_{ij}) + z_{ij}(E_{ii} + E_{ij}))$.*
2. *$C \in K_{prdd\text{-}\infty}$ if and only if there exists $y_{i,J} > 0$ for every pair (i, J) with $J \subseteq N$, $i \notin J$, such that $C = \sum_{i\notin J} y_{i,J}(\sum_{j\notin J} E_{ij} - \sum_{j\in J} E_{ij})$.*
3. *$C \in K_{Z^o-}$ if and only if there exist $y_{ij} > 0$ for $i, j \in N, i \neq j$ and $z_i > 0$ for all $i \in N$ such that $C = \sum_{i\neq j} y_{ij}(E_{ii} - E_{ij}) + \sum_{i=1}^{n} z_i E_{ii}$.*
4. *$C \in K_{Z^o+}$ if and only if there exist $y_{ij} > 0$ and $z_{ij} > 0$ for $i, j \in N, i \neq j$ such that $C = \sum_{i\neq j}(y_{ij}(E_{ii} - E_{ij}) + z_{ij}(-E_{ij}))$.*

Corollary 1 *Let $A \in \mathbb{R}^{n\times n}$.*

1. *Exactly one of the following holds:*
   (a) *A is hidden prdd.*
   (b) *There exist matrices R and S, not both 0, so that $R + A^T S = 0$, $R_{ii} \geq |R_{ij}|$, $S_{ii} \geq |S_{ij}|$ for all $i \neq j$.*

2. *Exactly one of the following holds:*
   (a) *A is hidden prdd-∞.*
   (b) *There exist matrices R and S, not both 0, so that $R + A^T S = 0$, and for every pair (i, G) with $G \subseteq N\backslash i$, $\sum_{j\notin G} R_{ij} - \sum_{j\in G} R_{ij} \geq 0, \sum_{j\notin G} S_{ij} - \sum_{j\in G} S_{ij} \geq 0$.*

3. *Exactly one of the following holds:*
   (a) *A is hidden Z^o-.*
   (b) *There exist matrices R and S, not both 0, so that $R + A^T S = 0$, $R_{ii} \geq R_{ij}$, $S_{ii} \geq S_{ij}$ for all $i \neq j$, $R_{ii} \geq 0$, $S_{ii} \geq 0$ for all i.*

4. *Exactly one of the following holds:*
   (a) *A is hidden Z^o+.*
   (b) *There exist matrices R and S, not both 0, so that $R + A^T S = 0$, $R_{ii} \geq R_{ij}$, $S_{ii} \geq S_{ij}$ for all $i \neq j$, $R_{ij} \leq 0$, $S_{ij} \leq 0$ whenever $i \neq j$.*

We will show later that the class hidden Z^o- is the same as the hidden Minkowski class. The dual nature of the classes hidden prdd and hidden prdd-∞ comes out in the following proposition, which can be read off from Corollary 1.

Proposition 2 *Let $A \in \mathbb{R}^{n\times n}$.*

1. *If $-A^T$ is hidden prdd, then A is not hidden prdd-∞.*
2. *If $-A^T$ is hidden prdd-∞, then A is not hidden prdd.*

Proof. A moment's reflection shows that the two statements are equivalent. If $-A^T$ is hidden prdd, then there exists (C, B) in $\mathbb{R}^{n\times 2n}$ so that $-A^T C = B$ and C and B are prdd. Because C and B are prdd, they satisfy the conditions for S and R to show that A is not hidden prdd-∞. ∎

The implications are not equivalences, because the conditions for non-membership in hidden prdd and hidden prdd-∞ involve non-strict inequalities, while the conditions for membership involve strict ones.

3 A P-matrix sandwich

An $n \times n$ matrix A is called a *P-matrix* if all its principal minors, that is, all determinants of principal submatrices, are positive. It is clear that if A is a P-matrix, then so are A^T and DAD, for any nonsingular diagonal matrix D.

Proposition 3 *The following are equivalent for $A \in \mathbb{R}^{n\times n}$:*

1. *A is a P-matrix.*
2. *For every nonzero $x \in \mathbb{R}^n$ there exists a subscript $i \in N$ so that $x_i(Ax)_i > 0$.*
3. *For every $G \subseteq N$, there exists a vector $x^G \in \mathbb{R}^n$ satisfying that $x_i^G < 0$ and $(Ax^G)_i < 0$ for $i \in G$, $x_j^G > 0$ and $(Ax^G)_j > 0$ for $j \notin G$.*

Proof. The equivalence of 1 and 2 is standard material, see [3]. Let C be the interior of the cone generated by the columns of the $n \times 2n$ matrix $(I, -A)$, and let M be the set $\{0\}$. Property 2, applied to A^T, implies that there is no hyperplane H through the origin that contains C in one of the open half-spaces associated with H. Theorem 1 therefore shows that 0 is in C, so there exist positive vectors y and x such that $y - Ax = 0$. This is the $G = \emptyset$ case of property 3. A matrix for which such x and y exist is called an *S-matrix.* For any $G \subseteq N$, let D^G be the diagonal matrix with $D_{ii}^G = -1$ for $i \in G$ and $D_{ii}^G = 1$ for $i \notin G$. Then $D^G A D^G$ is a P-matrix, so there exists a positive vector x^G so that $D^G A D^G x^G > 0$. This implies that $(D^G x^G)_i < 0$ and $(AD^G x^G)_i < 0$ for $i \in G$ and $(D^G x^G)_i > 0$ and $(AD^G x^G)_i > 0$ for $i \notin G$, so Property 3 holds. Now suppose Property 2 fails to hold. Then A is not a P-matrix, so A^T is not a P-matrix and Property 2 also fails for A^T. Let $x \neq 0$ satisfy $x_i(A^T x)_i \leq 0$ for all i. Let $G = \{i \in N | x_i < 0\}$. If there were a vector y so that $y_i < 0$ and $(Ay)_i < 0$ for all i in G and $y_i > 0, (Ay)_i > 0$ for all $i \notin G$, then we would have both $x^T(Ay) > 0$ and $(x^T A)y \leq 0$, a contradiction. ∎

Property 3 gives an exponentially large certificate for a matrix to be a P-matrix, but it is better than displaying 2^n determinant calculations. Note that one can always take $x^{N\backslash G} = -x^G$, so one only needs to display 2^{n-1} vectors to establish the P-property. We do not know if there is a certificate that is of size $O(k^n)$ for some $k < 2$.

Proposition 4 *Every P-matrix is a hidden prdd-∞ matrix.*

Proof. Suppose that a matrix $A \in \mathbb{R}^{n\times n}$ is not hidden prdd-∞. By Corollary 1 there exist matrices R and S, not both 0, so that $R = -A^T S$, and for every pair (i, G) with $G \subseteq N \backslash i$, $\sum_{j\notin G} R_{ij} - \sum_{j\in G} R_{ij} \geq 0, \sum_{j\notin G} S_{ij} - \sum_{j\in G} S_{ij} \geq 0$. At least one of these inequalities is strict, because R and S are not both 0. Let G^* correspond to one of the strict inequalities, say $\sum_{j\notin G^*} S_{ij} - \sum_{j\in G^*} S_{ij} > 0$ for some $i \notin G^*$. Then $(Se^{G^*})_j \geq 0$, $-(Re^{G^*})_j = (A^T S e^{G^*})_j \leq 0$ for $j \notin G^*$, $(Se^{G^*})_j \leq 0$, $-(Re^{G^*})_j = (A^T S e^{G^*})_j \geq 0$ for $j \in G^*$, and $Se^{G^*} \neq 0$, so A^T (and hence A) is not a P-matrix, by part (2) of Proposition 3. ∎

The following inclusion is known [2], but our proof is new and illustrates the usefulness of part 3 of Proposition 3.

Proposition 5 *Every hidden prdd matrix is a P-matrix.*

Proof. Let A be hidden prdd, that is, there exist prdd matrices C and B such that $AC = B$. Then one can see that $(Ce^G)_i < 0$ for $i \in G$, and $(Ce^G)_j > 0$ for $j \notin G$. The same holds for ACe^G, that is, $(ACe^G)_i < 0$ for $i \in G$, and $(ACe^G)_j > 0$ for $j \notin G$. The vector Ce^G may be used as x^G in Proposition 3. ∎

For any n, the number of extreme rays of the cone K_{prdd} is a quadratic polynomial in n, and the number of facets of the cone $K_{prdd\text{-}\infty}$ is a quadratic polynomial in n. Therefore, the problem of checking membership in the classes hidden prdd and hidden prdd-∞ has a polynomial time algorithm. This implies that one should not expect either of these classes to equal the class of P-matrices. It is easy to find 3×3 matrices that are hidden prdd-∞ but not P-matrices. It is harder, though, to find P-matrices that are not hidden prdd. We will show that a P-matrix that is not hidden prdd must be at least 4×4, and a symmetric and positive definite example of such a matrix was indeed found in [12].

4 Hidden k-prdd and hidden prdd-p matrices

We showed in the previous section that the class P is sandwiched between the classes hidden prdd-∞ and hidden prdd. It is therefore natural to try to enlarge the hidden prdd class or reduce the hidden prdd-∞ class to get closer to the class P. There exist many singular matrices, of the form $nI - e^G(e^G)^T$ with $G \subseteq N$, on the boundary of the set of prdd matrices. This makes it hard to find concisely defined matrix classes between hidden prdd and P. However, there could exist such a class between P and hidden prdd-∞. We search for such a class by defining a continuum of polyhedral cones that has prdd and prdd-∞ at its endpoints. As in the previous section, for $i, j \in N$, E_{ij} is the $n \times n$ matrix with element (i, j) equal to 1 and all other elements equal to 0.

Definition 3 (k-prdd and hidden k-prdd for real number k) *For a real number k with $1 \le k \le n-1$, a matrix $C \in \mathbb{R}^{n\times n}$ is called k-prdd if there exist positive numbers y_{ij}, z_{ij} for all i, j with $i \neq j$, and $w_{i,G}$ for every pair (i, G) with $G \subseteq N\backslash\{i\}$ so that $C = \sum_{i\neq j}(y_{ij}(E_{ii} - E_{ij}) + z_{ij}(E_{ii} + E_{ij})) + \sum_{i,G}\{w_{i,G}(E_{ii} + \sum_{j\notin G\cup\{i\}} \frac{1}{k}E_{ij} - \sum_{j\in G} \frac{1}{k}E_{ij})\}$.*

Moreover, a matrix A is called a hidden k-prdd matrix *if there exist k-prdd matrices C and B such that $AC = B$.*

Note that the matrices $E_{ii} + E_{ij}$ and $E_{ii} - E_{ij}$ for $i \neq j$ are the generators for the prdd cone. When $k = n-1$, each matrix $E_{ii} + \sum_{j\notin G\cup\{i\}} \frac{1}{k}E_{ij} - \sum_{j\in G} \frac{1}{k}E_{ij}$ for $i \notin G$ is a nonnegative combination of the generators of the prdd cone, and when $k = 1$, these matrices are the generators for the prdd-∞ cone (which contains the prdd cone). Thus 1-prdd is the same as prdd-∞ and $(n-1)$-prdd is the same as prdd. It is easy to check that for a fixed k, $1 < k < n-1$, none of these matrices is a nonnegative combination of the others.

Proposition 6 *If $1 \le k < l \le n-1$ and matrix $C \in \mathbb{R}^{n\times n}$ is l-prdd, then C is k-prdd.*

Proof. Suppose $C = \sum_{i\neq j}(y_{ij}(E_{ii}-E_{ij})+z_{ij}(E_{ii}+E_{ij}))+\sum_{i,G}\{w_{i,G}(E_{ii}+\sum_{j\notin G\cup\{i\}} \frac{1}{l}E_{ij} - \sum_{j\in G} \frac{1}{l}E_{ij})\}$ and define y'_{ij}, z'_{ij} for all i, j with $i \neq j$, and $w'_{i,G}$ for every pair (i, G)

with $G \subseteq N\backslash\{i\}$ by $w'_{i,G} = \frac{k}{l} w_{i,G}$ and $y'_{ij} = y_{ij} + \frac{1}{2(n-1)} \sum_{G\subseteq N\backslash\{i\}} w_{i,G}(1-\frac{k}{l})$, $z'_{ij} = z_{ij} + \frac{1}{2(n-1)} \sum_{G\subseteq N\backslash\{i\}} w_{i,G}(1-\frac{k}{l})$. Then $C = \sum_{i\neq j}(y'_{ij}(E_{ii}-E_{ij}) + z'_{ij}(E_{ii}+E_{ij})) + \sum_{i,G}\{w'_{i,G}(E_{ii} + \sum_{j\notin G\cup\{i\}} \frac{1}{k}E_{ij} - \sum_{j\in G}\frac{1}{k}E_{ij})\}$. ∎

The k-prdd cone is defined by a set of generators. We can therefore find its dual cone directly, by requiring a nonnegative inner product with each generator, and apply Theorem 2.

Proposition 7 (alternative theorem for hidden k-prdd matrices) *Let $A \in \mathbb{R}^{n\times n}$. Then exactly one of the following holds.*

(a) A is hidden k-prdd.

(b) There exist matrices R and S, not both 0, so that $R + A^TS = 0$, $R_{ii} \geq |R_{ij}|$, $S_{ii} \geq |S_{ij}|$ for all $i \neq j$, $R_{ii} \geq \frac{1}{k}\sum_{j\neq i}|R_{ij}|$, $S_{ii} \geq \frac{1}{k}\sum_{j\neq i}|S_{ij}|$ for all i.

Proposition 4 shows that every P-matrix is hidden 1-prdd. We will show later that every 3×3 P-matrix is hidden 2-prdd. As pointed out by a referee, it seems likely, because the cone $K_{prdd\text{-}\infty}$ is open, that every P-matrix A is hidden k-prdd for some $k > 1$ depending on A. On the other hand, Example 2 will show that when $n \geq 4$ there is no $k > 1$ so that every $n \times n$ P-matrix is hidden k-prdd.

We do not have an inequality description of the k-prdd cone for all $1 < k < n-1$. When k is a positive integer between 1 and $n-1$, however, the k-prdd cone has a particularly nice inequality description.

Proposition 8 *Let $C \in \mathbb{R}^{n\times n}$ and let k be an integer with $1 \leq k \leq n-1$. $C_{ii} > \sum_{j\in G}|C_{ij}|$ for all (i,G) such that $G \subseteq N\backslash\{i\}$ and $|G| = k$ if and only if C is k-prdd.*

Proof. We first prove the if part. Note that each of the generators $E_{ii} - E_{ij}$, $E_{ii} + E_{ij}$ for $i \neq j$ and $E_{ii} + \sum_{j\notin J\cup\{i\}}\frac{1}{k}E_{ij} - \sum_{j\in J}\frac{1}{k}E_{ij}$ for (i,J) with $i \notin J$ satisfies $C_{ii} \geq \sum_{j\in G}|C_{ij}|$ for all (i,G) such that $G \subseteq N\backslash\{i\}$ and $|G| = k$. Also, for every one of the inequalities there is at least one generator that satisfies it strictly. Therefore, every positive combination of the generators satisfies all of the inequalities strictly.

Now we show the only if part. For notational simplicity, for i and G with $G \subseteq N\backslash\{i\}$, let $e^{i,k,G}$ denote the transpose of the ith row of $E_{ii} + \sum_{j\notin G\cup\{i\}}\frac{1}{k}E_{ij} - \sum_{j\in G}\frac{1}{k}E_{ij}$ or, equivalently, $e^{i,k,G} = e_i + \sum_{j\notin G\cup\{i\}}\frac{1}{k}e_j - \sum_{j\in G}\frac{1}{k}e_j$. It suffices to show that, for fixed i, if $C_{ii} > \sum_{j\in G}|C_{ij}|$ for all G such that $G \subseteq N\backslash\{i\}$ and $|G| = k$ then $C^T_{i\cdot}$ is a positive combination of vectors $e_i - e_j$, $e_i + e_j$ and $e^{i,k,G}$ with $j \neq i$ and $G \subseteq N\backslash\{i\}$. We show this by induction on the number $n-k$. For the basis of induction, we consider the case $n-k = 1$, that is $C_{ii} > \sum_{j\neq i}|C_{ij}|$. Let us define $\varepsilon = C_{ii} - \sum_{j\neq i}|C_{ij}| > 0$, $y_{ij} = -C_{ij}$ if $C_{ij} < 0$, $y_{ij} = 0$ otherwise; $z_{ij} = C_{ij}$ if $C_{ij} > 0$, $z_{ij} = 0$ otherwise; and $w_{i,G} = 0$ for $G \subseteq N\backslash\{i\}$. Then $C^T_{i\cdot} = \varepsilon e_i + \sum_{j\neq i} y_{ij}(e_i - e_j) + \sum_{j\neq i} z_{ij}(e_i + e_j) + \sum_{G\subseteq N\backslash\{i\}} w_{i,G}e^{i,(n-1),G}$ holds. Moreover, let $y'_{ij} = y_{ij} + \frac{1}{4(n-1)}\varepsilon$, $z'_{ij} = z_{ij} + \frac{1}{4(n-1)}\varepsilon$ and $w'_{i,G} = w_{i,G} + \frac{1}{2^n}\varepsilon$. Then we have: $C^T_{i\cdot} = \sum_{j\neq i} y'_{ij}(e_i - e_j) + \sum_{j\neq i} z'_{ij}(e_i + e_j) + \sum_{G\subseteq N\backslash\{i\}} w'_{i,G}e^{i,(n-1),G}$ because $e_i = \frac{1}{2(n-1)}\{\sum_{j\neq i}(e_i - e_j) + \sum_{j\neq i}(e_i + e_j)\} = \frac{1}{2^{n-1}}\sum_{G\subseteq N\backslash\{i\}} e^{i,(n-1),G}$. Furthermore, all of the variables $y'_{ij}, z'_{ij}, w'_{i,G}$ are strictly positive.

For the case $n-k > 1$, suppose that the statement holds for any n' and k' such that $1 \leq n'-k' < n-k$ and let $C \in \mathbb{R}^{n\times n}$ satisfy $C_{ii} > \sum_{j\in G}|C_{ij}|$ for all G satisfying

$G \subseteq N\backslash\{i\}$ and $|G| = k$ with $1 \leq k < n-1$. One can take sufficiently small $\varepsilon > 0$ such that $C_{ii} - \varepsilon > \sum_{j\in G} |C_{ij}|$ for all $G \subseteq N\backslash\{i\}$ with $|G| = k$. Let $r \in N\backslash\{i\}$ satisfy $|C_{ir}| = \min_{j\neq i} |C_{ij}|$ and let $J = \{j \neq i : C_{ij} < 0\}$. Then we define $C'^T_{i\cdot} = C^T_{i\cdot} - \varepsilon e_i - k|C_{ir}|e^{i,k,J}$ so that $C'_{ir} = 0$. It is straightforward to check that $C'_{ii} > \sum_{j\in G} |C'_{ij}|$ for all $G \subseteq N\backslash\{i, r\}$ with $|G| = k$. Let $C''^T_{i\cdot}$ be the subvector of $C'^T_{i\cdot}$ with element r removed. From the inductive hypothesis, there exist $y'_{ij} > 0$ and $z'_{ij} > 0$ for $j \neq i, r$ and $w'_{i,G} > 0$ for $G \subseteq N\backslash\{i, r\}$ so that $C''^T_{i\cdot} = \sum_{j\neq i,r}(y'_{ij}(e'_i - e'_j) + z'_{ij}(e'_i + e'_j)) + \sum_{G\subseteq N\backslash\{i,r\}} w'_{i,G}(e'^{i,k,G})$, where the e'_j and $e'^{i,k,G}$ vectors are the corresponding subvectors of e_j and $e^{i,k,G}$ with element r removed. Let $y_{ij} = y'_{ij}$, $z_{ij} = z'_{ij}$ for $j \neq i, r$, $y_{ir} = \frac{1}{2}\varepsilon$, $z_{ir} = \frac{1}{2}\varepsilon$, and let $w_{i,G\cup\{r\}} = \frac{1}{2}w'_{i,G}$, $w_{i,G} = \frac{1}{2}w'_{i,G}$ for $G \subseteq N\backslash\{i, r\}$. We have: $C^T_{i\cdot} = \sum_{j\neq i}(y_{ij}(e_i - e_j) + z_{ij}(e_i + e_j)) + \sum_{G\subseteq N\backslash\{i\}} w_{i,G}e^{i,k,G} + k|C_{ir}|e^{i,k,J}$ because $e_i = \frac{1}{2}\{(e_i - e_r) + (e_i + e_r)\}$ and $\sum_{G\subseteq N\backslash\{i\}}(e^{i,k,G})_r = 0$. Since $J \subseteq N\backslash\{i\}$, this shows that $C^T_{i\cdot}$ is a positive combination of $e_i - e_j, e_i + e_j$ and $e^{i,k,G}$ for all $j \neq i$ and all $G \subseteq N\backslash\{i\}$. This completes the proof. ∎

When $k = \lfloor \frac{n}{2} \rfloor$, both the number of facets and the number of extreme points of the k-prdd cone can be shown to be exponential functions of n. Therefore, membership in the hidden k-prdd class probably cannot be verified efficiently for such k.

Note that a matrix $C \in \mathbb{R}^{n\times n}$ is prdd if and only if $C_{ii} > \|C_{i\cdot} - C_{ii}e_i\|_1$ and C is prdd-∞ if and only if $C_{ii} > \|C_{i\cdot} - C_{ii}e_i\|_\infty$, for all i. It is therefore natural to extend the definition to other p-norms.

Definition 4 (prdd-p and hidden prdd-p) *A matrix $C \in \mathbb{R}^{n\times n}$ is called a* prdd-p, *or a* prdd-p matrix *if it satisfies: $C_{ii} > (\sum_{j\neq i} |C_{ij}|^p)^{1/p}$ for all $i \in N$. Moreover, a matrix $A \in \mathbb{R}^{n\times n}$ is called* hidden *prdd-p if there exist prdd-p matrices C and B such that $AC = B$.*

The next proposition is based on Theorem 2 and the fact (see [13], pp. 135-136) that the closure of the prdd-p cone is dual to the closure of the prdd-q cone, where p and q are real numbers with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 9 *Let $A \in \mathbb{R}^{n\times n}$ and let p and q be real numbers with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then exactly one of the following holds.*

(a) A is hidden prdd-p.

(b) There exist matrices R and S, not both 0, so that $R + A^TS = 0$, and for each i $R_{ii} \geq (\sum_{j\neq i} |R_{ij}|^q)^{1/q}$, $S_{ii} \geq (\sum_{j\neq i} |S_{ij}|^q)^{1/q}$.

It is natural to ask if there is a finite number $p > 1$ so that every P-matrix is prdd-p. The P-matrices that appear in Example 2 are all hidden prdd-2. We have not found any P-matrices that are not hidden prdd-2. For this reason, we make the following conjecture.

Conjecture 1 *Every P-matrix is a hidden prdd-2 matrix.*

Membership in the hidden prdd-2 class can be verified using second-order cone programming. For a recent survey of second-order cone programming, see [14].

5 Hidden Z^o-matrices and a second P-matrix sandwich

Recall that a matrix is a Z-matrix if its off-diagonal entries are non-positive. The Z-matrices that interest us have nonnegative diagonals. By also requiring that the entries all be nonzero, we get a class of matrices we call Z^o.

Definition 5 (Z^o-matrices and hidden Z^o-matrices) *A square matrix A is called a Z^o-matrix if its off-diagonal entries are negative and its diagonal entries are positive. It is called a hidden Z^o-matrix if there exist Z^o-matrices C and B so that $AC = B$.*

Lemma 1 *Every P-matrix A is a hidden Z^o-matrix.*

Proof. Let A be a P-matrix. By Proposition 3, there exist vectors x^i so that $x^i_i > 0$ and $(Ax^i)_i > 0$ for all i and $x^i_j < 0$ and $(Ax^i)_j < 0$ for $j \neq i$. Form the matrix C by letting its column i be x^i, $i = 1, \ldots, n$. Then C is a Z^o-matrix. For each i, column i of AC is Ax^i, so AC is also a Z^o-matrix. ∎

Our next theorem will show that every P-matrix is a hidden Z^o+-matrix. The main step is to show that a matrix is hidden Z^o+ if and only if it is both a hidden Z^o-matrix and an S-matrix, i.e. if property (3) of Proposition 3 holds for all sets G of size $n-1$ and 0.

Lemma 2 *Let A be a hidden Z^o and S-matrix. Then there exist Z^o- matrices C', B' and a vector $x' > 0$ such that $AC' = B'$, C'^{-1} exists, $Ax' > 0$ and $C'^{-1}x' < 0$.*

Proof. Let A be a hidden Z^o and S-matrix with Z^o-matrices C, B and a vector $x > 0$ satisfying $AC = B$ and $Ax > 0$. Because all entries of C and B are nonzero, we can perturb C if necessary so that C^{-1} exists, C and AC remain Z^o-matrices and $C^{-1}x$ has all nonzero components. Let y solve $Cy = x$. Consider the sign pattern of y.

If $y < 0$, obviously C, B and x satisfy the requirements.

If $y > 0$, define $C(t) := C - xte^T$ and $y(t) := \frac{1}{1-te^Ty}y$ for $t > 0$. Let t_1 be the smallest t for which a diagonal element of $C(t)$ or $AC(t)$ is 0, and let t_2 satisfy $1 - t_2e^Ty = 0$. Note that for $t \neq t_2$, $C(t)y(t) = (\frac{1}{1-te^Ty})(C - xte^T)y = x$ and that for $t < t_2$, $y(t) > 0$ and for $t > t_2$, $y(t) < 0$ holds. If a diagonal element i of $C(t_1)$ is 0, then $t_1 > t_2$ must hold since $x_i = (C(t_1)y(t_1))_i > 0$. If a diagonal element i of $AC(t_1)$ is 0, then $t_1 > t_2$ must hold again because $(Ax)_i = (AC(t_1)y(t_1))_i > 0$. Let t_3 be between t_1 and t_2 such that $C(t_3)$ is invertible. Then $C' = C(t_3)$, $B' = AC(t_3)$ and $x' = x$ satisfy the requirements of the lemma.

If $y \not> 0$ and $y \not< 0$, let $F = \{i|y_i > 0\}$, $G = \{i|y_i < 0\}$ and define $x(\varepsilon) := x + \sum_{i\in F} C_{\cdot i}(-\varepsilon - y_i)$ for $\varepsilon > 0$. For $j \in F$, $x(\varepsilon)_j = (Cy)_j - \sum_{i\in F} C_{ji}y_i - \sum_{i\in F} \varepsilon C_{ji} = \sum_{i\in G} C_{ji}y_i - \sum_{i\in F} \varepsilon C_{ji} > 0$ for ε small enough. Also, $(Ax(\varepsilon))_j = (ACy)_j - \sum_{i\in F}(AC)_{ji}y_i - \sum_{i\in F} \varepsilon(AC)_{ji} = \sum_{i\in G}(AC)_{ji}y_i - \sum_{i\in F} \varepsilon(AC)_{ji} > 0$ for ε small enough. Finally, $(C^{-1}x(\varepsilon))_j = y_j - \varepsilon - y_j < 0$. Now consider $j \in G$. Then $x(\varepsilon)_j = x_j + \sum_{i\in F}(C_{ji}(-\varepsilon - y_i)) > 0$ because the summands are all positive. Similarly, $(Ax(\varepsilon))_j = (Ax)_j + \sum_{i\in F}(AC)_{ji}(-\varepsilon - y_i) > 0$. Also, $(C^{-1}x(\varepsilon))_j = y_j - 0 < 0$. Thus $C' = C$, $B' = B$ and $x' = x(\varepsilon)$ satisfy the requirements, for $\varepsilon > 0$ small enough. ∎

Lemma 3 *Let A be a hidden Z^o and S-matrix. Then A is a hidden Z^o+-matrix.*

Proof. Suppose C', B' and x' are as in Lemma 2. Let y' satisfy $C'y' = x'$. Then define $C = C'\mathrm{diag}(-y')$ and $B = AC$. The matrices B and C are still Z^o-matrices, and $Ce = -C'y' < 0$, $Be = -AC'y' < 0$. ■

Lemmas 1 and 3, together with Proposition 3, imply the following theorem.

Theorem 3 *Every P-matrix is a hidden Z^o+-matrix.*

Section 7 contains a 4×4 matrix that is a hidden Z^o+-matrix, but is not a P-matrix. It seems plausible that the class of P-matrices can be contained in hidden matrix classes defined by ever larger subsets of the prdd inequalities, as n increases. However, judging by the difficulty we had in going from the hidden Z^o case to the hidden Z^o+ case, even the next step seems difficult. Perhaps a reader can find an easier proof to show that P-matrices are hidden Z^o+.

Lemma 4 *Let C be a Z^o+-matrix in $\mathbb{R}^{n\times n}$, with $n \geq 2$. Then $C(I - ee^T)$ is a prdd-∞ matrix. If $n = 3$, then $C(I - ee^T)$ is a prdd matrix. If $n = 2$, then $C(I - ee^T)$ is a Z^o- matrix.*

Proof. The generators of the Z^o+ cone are of the form $E_{ii} - E_{ij}$ and $-E_{ij}$ for $i \neq j$. The images of these generators are $(E_{ii} - E_{ij})(I - ee^T) = E_{ii} - E_{ij}$ and $-E_{ij}(I - ee^T) = \sum_{k\neq j} E_{ik}$. Every positive linear combination of these vectors satisfies all of the inequalities defining the prdd-∞ cone. If $n = 3$, then the images are the generators of the prdd cone, and for $n = 2$ they generate the Z^o- cone. ■

Corollary 2 *Let A be a hidden Z^o+-matrix in $\mathbb{R}^{n\times n}$. Then A is a hidden prdd-∞ matrix. If $n = 3$, then A is hidden prdd, and if $n = 2$, then A is hidden Z^o-.*

Theorem 4 *Let $A \in \mathbb{R}^{3\times 3}$. Then the following four statements are equivalent:*

- *(a) A is a hidden prdd matrix.*
- *(b) A is a P-matrix.*
- *(c) A is a hidden Z^o-matrix and an S-matrix.*
- *(d) A is a hidden Z^o+-matrix.*

Proof. For any matrix $A \in \mathbb{R}^{n\times n}$, Proposition 5 shows (a) $\Rightarrow$ (b), Lemma 1 and the necessary condition for A to be a P-matrix show (b) $\Rightarrow$ (c), Lemma 3 shows (c) $\Rightarrow$ (d). For a $A \in \mathbb{R}^{3\times 3}$, Corollary 2 shows (d) $\Rightarrow$ (a). ■

A Minkowski matrix is a Z-matrix that is also an S-matrix. Recall that a matrix A is an S-matrix if there is a vector $x > 0$ so that $Ax > 0$. Unfortunately, the class known in the literature (see [7]) as hidden Minkowski is not the set of matrices A for which there exist Minkowski matrices C and B so that $AC = B$. We will show, however, that it is the set of matrices A for which there exist Z^o--matrices B and C so that $AC = B$. Proposition 6.1 of Tsatsomeros [2] showed the following to be equivalent to earlier definitions of the hidden Minkowski class, so we take it as our definition.

Definition 6 (Hidden Minkowski matrix) *A matrix $A \in \mathbb{R}^{n\times n}$ is called a* hidden Minkowski *matrix if there exist Z-matrices C and $B \in \mathbb{R}^{n\times n}$ so that $AC = B$ and both A and C are S-matrices.*

The following is a strengthening of Proposition 6.4 of [2].

Proposition 10 *A matrix $A \in \mathbb{R}^{n\times n}$ is a hidden Minkowski matrix if and only if there exist Z^o--matrices C and B so that $AC = B$.*

Proof. Suppose A is hidden Minkowski. Let B and C be Z-matrices so that $AC = B$ and let x and v in $\mathbb{R}^n$ satisfy $x > 0$, $Ax > 0$, $v > 0$, $Cv > 0$. Then C is a Minkowski matrix, hence it has a nonnegative inverse. Let $u = C^{-1}x$. Then $u > 0$, $Cu = x > 0$ and $Bu = ACu = Ax > 0$. Let U be the diagonal matrix with the entries of u on its diagonal. Then $CU((n+1)I - ee^T)$ and $BU((n+1)I - ee^T)$ are Z^o-matrices, and $CU((n+1)I - ee^T)e = nCUe > 0, BU((n+1)I - ee^T)e = nBUe > 0$.

To prove the converse, let B and C be Z^o-matrices so that $AC = B$ and $Be > 0$, $Ce > 0$. Let $x = Ce$. Then $Ax = ACe = Be > 0$, so A and C are S-matrices.

∎

This means that the hidden Minkowski matrices are the same as the hidden Z^o--matrices defined earlier, and the alternative theorem for that class applies.

The numbers of facets of the cones K_{Z^o+} and K_{Z^o-} are both quadratic functions of n. Therefore, membership in the hidden Z^o+ and hidden Z^o- classes can be determined in polynomial time. Note that the cone of Z^o--matrices is properly contained in the cone of *prdd*-matrices. Therefore, the hidden Minkowski matrices are P-matrices. Together with Theorem 3 and Corollary 2, this gives us our final good P-matrix sandwich: hidden Z^o- $\subseteq$ hidden prdd $\subseteq$ P $\subseteq$ hidden Z^o+ $\subseteq$ hidden prdd-∞.

The following relative of Lemma 1, that P-matrices are hidden Z^o, already appeared in [8], but we include it for completeness.

Theorem 5 *Let A be a P-matrix. Then there exist unique Z-matrices C and B with ones on the main diagonal of C and positive entries on the main diagonal of B, so that $AC = B$ and $C_{ij}B_{ij} = 0$ whenever $i \neq j$. Furthermore, A is hidden Minkowski if and only if C is an S-matrix.*

We close this section with some observations on linear complementarity problems. Recall that the linear complementarity problem with matrix A in $\mathbb{R}^{n\times n}$and vector q in $\mathbb{R}^n$ is to find nonnegative vectors x and y so that $y - Ax = q$ and $x^T y = 0$. We can append the $n \times n$ identity matrix to the bottom of A to obtain a $2n \times n$ matrix A' with a natural partitioning of the rows into n two element sets $\{A_{i.}, e_i^T\}$. We can also append the zero vector of length n to q to get a vector q' of length $2n$. The LCP can then be reformulated as seeking a vector x so that $A'x + q' \geq 0$, and with equality holding for at least one row in each of the n two-element sets.

For an $A' \in \mathbb{R}^{m\times n}$ and $q' \in \mathbb{R}^n$ and a partition $(I_1, I_2, \ldots, I_n)$ of $\{1, 2, \ldots, m\}$, the generalized LCP is to find $x \in \mathbb{R}^n$ for which $A'x + q' \geq 0$ and with $\prod_{i\in I_j}(A'x + q')_i = 0$ for $j = 1, 2, \ldots, n$. Following Mangasarian [15], we do not require that, for each $j \in \{1, 2, \ldots, n\}$ there be an index $i \in I_j$ so that $A'_{i.} = e_j^T$ and $q'_i = 0$. Note that if, for each

$j \in \{1, 2, \ldots, n\}$, we have $|I_j| = 2$ and there is an $i \in I_j$ so that $A'_{i\cdot} = e_j^T$ and $q'_i = 0$, then the generalized LCP is a usual LCP. An $n \times n$ submatrix of A' formed by taking, for each j, row j to be one of the rows indexed by I_j, is called a *representative* submatrix of A'. A fundamental insight (reformulated in our terms) of Mangasarian [6] is that if each representative submatrix of A' is a prdd Z-matrix, then the generalized LCP can be solved by solving the linear program: minimize $e^T x$ subject to $A'x + q' \geq 0$. If it is not true that every representative submatrix of A' is a prdd Z-matrix but there is an $n \times n$ matrix C so that every representative submatrix X of $A'C$ is a prdd Z-matrix, (as is the case in the usual LCP when A is a hidden Z^o- matrix), then every such X is a P-matrix by Proposition 5, so C is nonsingular and the generalized LCP can be solved by minimizing $e^T C^{-1} x$ subject to $A'x + q' \geq 0$.

If we require the representative submatrices of A' to be in the *prdd* cone, rather than in the smaller cone of prdd Z-matrices, the linear program of Mangasarian will not necessarily solve the generalized LCP. Theorem 11 of the paper of Svensson and Vorobyov [11] states that every stopping simple stochastic game reduces to an instance of the D-matrix generalized LCP, which is a generalized LCP for which the representative submatrices of A' are all prdd and nonnegative entrywise, and with some extra block structure.

6 Inverses of prdd matrices

The following known properties were of use to us in generating one of the examples of the next section.

Proposition 11 *Let $C \in \mathbb{R}^{n \times n}$ be a prdd matrix and $M \in \mathbb{R}^{n \times n}$ be the matrix with $M_{ii} = C_{ii}$, $M_{ij} = -|C_{ij}|$ for all $i \neq j$. Then C^{-1} and M^{-1} exist and the following inequalities hold: $C^{-1}_{ii} > |C^{-1}_{ji}|$, $M^{-1}_{ii} > M^{-1}_{ji} \geq 0$ for $i, j \in N, (i \neq j)$.*

Proof. Proposition 5 shows that C and M are P-matrices. Therefore C^{-1} and M^{-1} exist and have positive diagonals. The first inequality is due to Horn and Johnson [16], the second is due to Metzler [17]. We give a proof that shows how the results relate to the cones we have been using in this paper. For the first inequality, we define an open cone in $\mathbb{R}^n$ with respect to $i \in N$: $K^{(i)}_{prdd} := \{x \in \mathbb{R}^n | x_i > \sum_{j \neq i} |x_j|\}$, which can be considered as a restriction of K_{prdd} associated with a fixed i. We also define $K^{(i)\#}_{prdd} := \{y \in \mathbb{R}^n | x^T y > 0 \text{ for all } x \in K^{(i)}_{prdd}\}$. Then it is easily seen that $K^{(i)\#}_{prdd} = \{y \in \mathbb{R}^n | y_i \geq |y_j| \text{ for all } j \neq i\} \backslash \{0\}$. Now, we will show $C^{-1}_{ii} > |C^{-1}_{ji}|$ for all $j \neq i$. Suppose the contrary, that is, there exist $i \neq j$ such that $C^{-1}_{ii} \leq |C^{-1}_{ji}|$. Let r be an index that attains $\max_{k \neq i} |C^{-1}_{ki}|$. If $C^{-1}_{ri} > 0$, then i^{th} column vector $C^{-1}_{\cdot i}$ is in the cone $K^{(r)\#}_{prdd}$. This implies that $C_{r\cdot} C^{-1}_{\cdot i} > 0$. If $C^{-1}_{ri} < 0$, then $-C^{-1}_{\cdot i} \in K^{(r)\#}_{prdd}$. This implies that $C_{r\cdot}(-C^{-1}_{\cdot i}) > 0$. However, these contradict $C_{j\cdot} C^{-1}_{\cdot i} = 0$ for all $j \neq i$.

Similarly, we can prove inequalities $M^{-1}_{ii} > M^{-1}_{ji} \geq 0$ using the cones $K^{(i)}_{Z^o-} := \{x \in \mathbb{R}^n | x_j < 0 \text{ for } j \neq i \text{ and } x^T e > 0\}$ and $K^{(i)\#}_{Z^o-} = \{y \in \mathbb{R}^n | y_i \geq 0 \text{ and } y_i \geq y_j\} \backslash \{0\}$. ∎

It seems difficult to generalize Proposition 11 to matrix classes that properly include prdd, because of the singular matrices on the boundary of the prdd cone. We have the following proposition for the case $n \leq 3$.

Proposition 12 *For $n \leq 3$, let C be an $n \times n$ Z^o+-matrix. Then C is invertible and C^{-1} is a Z-matrix and satisfies $C^{-1}_{ii} > C^{-1}_{ji}$ for $i, j = 1, 2, 3$ ($i \neq j$).*

Proof. For $n \leq 2$, the statement is obvious. Let $n = 3$. From Lemma 4, there exists a 3×3 prdd matrix C' such that $C = C'(2I - ee^T)$. This implies that C^{-1} exists and $C^{-1} = -\frac{1}{2}(ee^T - I)C'^{-1}$. From Proposition 11, $C'^{-1}_{ii} > 0$ and $C'^{-1}_{ii} > |C'^{-1}_{ij}|$ implies that $C^{-1}_{ij} < 0$ for $j \neq i$. Once we get the Z-matrix C^{-1}, we can prove that $C^{-1}_{ii} > C^{-1}_{ji}$ for $i, j = 1, 2, 3 (i \neq j)$ similarly to the proof of Proposition 11. ∎

7 Examples

In this section, we will illustrate the usefulness of the alternative theorems by giving short proofs of membership in classes discussed in earlier sections.

Example 1 *We first give a short proof that the 3×3 positive definite matrix $A = \begin{bmatrix} 1 & -1 & 20 \\ -1 & 10 & -23 \\ 20 & -23 & 402 \end{bmatrix}$ given in [9], is not a hidden Minkowski matrix.*

Proof. $R = \begin{bmatrix} 354 & 147 & 354 \\ 0 & 0 & -207 \\ 6982 & 2842 & 6982 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1127 & -1334 \\ -46 & 0 & 0 \\ -20 & 49 & 49 \end{bmatrix}$ as in part (1) of Corollary 1 immediately show that A is not hidden Minkowski.

∎

Example 2 *For any $n \geq 4$, consider the $n \times n$ matrix $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$, where $A = \frac{1}{2+t^2}\begin{bmatrix} t^2 & 0 & 2 & 2t \\ 0 & t^2 & -2t & 2 \\ -2 & 2t & t^2 & 0 \\ -2t & -2 & 0 & t^2 \end{bmatrix}$ is the upper left corner. Then $\overline{A}$ is positive definite for every $t > 0$. If $0 < t \leq 1$, then $\overline{A}$ is not hidden $(1+t)$-prdd.*

Proof. Let $\overline{S}$ be the $n \times n$ matrix $\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, where $S = \begin{bmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & t & -1 \\ 1 & -t & 1 & 0 \\ t & 1 & 0 & 1 \end{bmatrix}$, and let $\overline{R} = \overline{S}^T$. Then $\overline{R} + \overline{A}^T\overline{S} = 0$ and $\overline{A}$ is not hidden $(1+t)$-prdd.

On the other hand, with $C = \begin{bmatrix} 1 & 0 & -1+t^2 & -t \\ 0 & 1 & t & -1+t^2 \\ 1-t^2 & -t & 1 & 0 \\ t & 1-t^2 & 0 & 1 \end{bmatrix}$, one sees that both C and AC are prdd-2, so that A is hidden prdd-2. ∎

Example 3 *Let A be the 4×4 matrix:*

$$A = \frac{1}{t^2+1}\begin{bmatrix} t^2-1 & 0 & -\sqrt{2}t & \sqrt{2}t \\ 0 & t^2-1 & -\sqrt{2}t & -\sqrt{2}t \\ \sqrt{2}t & \sqrt{2}t & t^2-1 & 0 \\ -\sqrt{2}t & \sqrt{2}t & 0 & t^2-1 \end{bmatrix}.$$

A is positive definite for any $t > 1$. If $1 < t \le \sqrt{2}$ then A is not hidden prdd-p for $1 \le p \le \frac{2\log 2}{\log 2 + 2\log t}$. On the other hand, A is hidden prdd-2 for any $t > 1$.

Proof. Let $X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$. Then, $X^T X = I$ and $X = -X^T$ hold. Let $R = I - tX$ and $S = I + tX$. Then $RS = (I - tX)(I + tX) = (t^2 + 1)I$, so $-RS^{-1} = -\frac{1}{t^2+1}(I - tX)^2 = \frac{1}{t^2+1}\{(t^2-1)I + 2tX\} = A^T$. For $1 < t < \sqrt{2}$, we can see that $1 \ge \{(\frac{1}{\sqrt{2}}t)^q + (\frac{1}{\sqrt{2}}t)^q\}^{\frac{1}{q}}$ with $2 < \frac{2\log 2}{\log 2 - 2\log t} \le q < +\infty$. This shows that A is not hidden prdd-p for $1 < p \le \frac{2\log 2}{\log 2 + 2\log t} < 2$. When we take $t = \sqrt{2}$, we get a positive definite matrix A which is not hidden prdd. Witnesses are $R = I - \sqrt{2}X$ and $S = I + \sqrt{2}X$. Thus we obtain the first statement.

On the other hand, let $C = tI - X^T$ and $B = tI + X^T$. Then $CB = (tI - X^T)(tI + X^T) = (t^2 + 1)I$, so $BC^{-1} = \frac{1}{t^2+1}(tI + X^T)^2 = \frac{1}{t^2+1}\{(t^2 - 1)I + 2tX^T\} = A$. Clearly, for any $t > 1$, C and B are prdd-2 matrices.

The above construction can be extended for $n > 4$. Consider the $n \times n$ matrix $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$, where A is the matrix defined above and is the upper left corner. Then $\overline{A}$ is positive definite for every $t > 1$. If $1 < t \le \sqrt{2}$, then $\overline{A}$ is not hidden prdd-p with $1 \le p \le \frac{2\log 2}{\log 2 + 2\log t} < 2$. Moreover $\overline{A}$ is a hidden prdd-2 matrix for any $t > 1$. Extended witnesses are the matrices $\overline{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$, $\overline{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, $\overline{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$ and $\overline{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$. ∎

The next example gives a construction technique to create, for odd dimension, a positive definite matrix that is not hidden prdd.

Example 4

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

The matrix A is not hidden prdd, certified by matrices R and S as in Corollary 1.

$$R = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}, S = \begin{bmatrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

On the other hand, we can show that A is "almost" hidden prdd. Let

$$C = \begin{bmatrix} 6+t & -4 & 0 & 2 & 0 \\ 0 & 6+t & -4 & 0 & 2 \\ 2 & 0 & 6+t & -4 & 0 \\ 0 & 2 & 0 & 6+t & -4 \\ -4 & 0 & 2 & 0 & 6+t \end{bmatrix}, B = AC = \begin{bmatrix} 8+t & 2+t & 2+t & -2 & 2 \\ 2 & 8+t & 2+t & 2+t & -2 \\ -2 & 2 & 8+t & 2+t & 2+t \\ 2+t & -2 & 2 & 8+t & 2+t \\ 2+t & 2+t & -2 & 2 & 8+t \end{bmatrix}.$$

The matrix C is clearly prdd for $t > 0$, and for $0 < t < 8-2k$ we have

$$B = 2k \begin{bmatrix} 1 & \frac{1}{k} & \frac{1}{k} & \frac{-1}{k} & \frac{1}{k} \\ \frac{1}{k} & 1 & \frac{1}{k} & \frac{1}{k} & \frac{-1}{k} \\ \frac{-1}{k} & \frac{1}{k} & 1 & \frac{1}{k} & \frac{1}{k} \\ \frac{1}{k} & \frac{-1}{k} & \frac{1}{k} & 1 & \frac{1}{k} \\ \frac{1}{k} & \frac{1}{k} & \frac{-1}{k} & \frac{1}{k} & 1 \end{bmatrix} + t \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix} + (8-2k-t) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This shows that A is hidden k-prdd for every $k < 4$, because the identity matrix is in the interior of the k-prdd cone.

Example 5 *Let $A = \begin{bmatrix} -1 & -5 & 1 & 1 \\ -5 & -1 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix}$. The matrix A is hidden prdd-2 and hidden Z^o+ , but A is neither a P-matrix nor even a Q-matrix (There exists a q so that the LCP with matrix A and vector q has no solution).*

Proof. Let $C = \begin{bmatrix} 5 & -1 & 1 & 1 \\ -1 & 5 & 1 & 1 \\ 8 & 8 & 16 & -8 \\ 8 & 8 & -8 & 16 \end{bmatrix}$ and $B = \begin{bmatrix} 16 & -8 & 2 & 2 \\ -8 & 16 & 2 & 2 \\ 100 & 100 & 170 & -70 \\ 100 & 100 & -70 & 170 \end{bmatrix}$ satisfy $AC = B$ and show that A is hidden prdd-2. Let $C = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -8 \\ -1 & -1 & -8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -6 & -1 & -1 \\ -6 & 2 & -1 & -1 \\ -12 & -12 & 1 & -89 \\ -12 & -12 & -89 & 1 \end{bmatrix}$. Then $AC = B$, C and B are Z^o-matrices, and for $x = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$, we have $Ax = \begin{bmatrix} 2 \\ 2 \\ 50 \\ 50 \end{bmatrix}$. A is obviously not a P-matrix. The linear complementarity problem with matrix A and right-hand side vector $q = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ has no solution.

To see this, note that if $y - Ax = q$ is to have a solution in nonnegative x and y, then x_3 or x_4 must be positive, because the first two entries of q are negative. Assume $x_3 > 0$. The third entry of column 3 of $-A$ is more negative than the first two entries of the column, so this must be balanced by using a column of $(I, -A)$ in which the third entry is larger than the first two entries. The only such column is the third column of I, and using it would violate the complementarity condition $x^T y = 0$. Therefore A is not a Q-matrix. We used the program TOPCOM, by J. Rambau, to help us find a vector q for which the LCP had no solution. ∎

Example 6 *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -2 & 1 & 1 & -1 \\ -2 & -2 & 1 & 2 \\ -2 & 0 & -1 & 1 \end{bmatrix}$. *Then A is hidden Minkowski but A^T is not even hidden prdd.*

Proof. The following matrices C and B show, according to Proposition 10, that A is a hidden Minkowski matrix. Note that these matrices also satisfy the relationship $C_{ij}B_{ij} = 0$ as in Theorem 5.

$$C = \begin{bmatrix} 22 & -21 & 0 & 0 \\ 0 & 105 & -99 & 0 \\ 0 & 0 & 99 & -96 \\ 0 & -42 & 0 & 48 \end{bmatrix}, B = \begin{bmatrix} 22 & 0 & 0 & 0 \\ -44 & 189 & 0 & -144 \\ -44 & -252 & 297 & 0 \\ -44 & 0 & -99 & 144 \end{bmatrix}.$$

The matrices R and S defined as follows satisfy $R + AS = 0$, $R_{ii} \geq |R_{ij}|$ and $S_{ii} \geq |S_{ij}|$ for $i, j = 1, 2, 3, 4$, which shows that A^T is not hidden prdd by Corollary 1.

$$S = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, R = \begin{bmatrix} 3 & -3 & 0 & 3 \\ 3 & 3 & -3 & 3 \\ 3 & 3 & 3 & -3 \\ 2 & 0 & 2 & 2 \end{bmatrix}.$$

∎

Example 6 was obtained by numerical experiments based on Proposition 11, which says that every hidden prdd matrix A can be decomposed into a product of the inverse of a prdd matrix C^{-1} with the property $C^{-1}_{ii} > |C^{-1}_{ji}|$ for $j \neq i$ and a prdd matrix B so that $A = C^{-1}B$. In order to find matrices as in Example 6, we randomly generated matrices $A = C'B$ such that C' is a matrix with $C'_{ii} > |C'_{ji}|$ and B is a prdd matrix. We found a few matrices that are hidden Minkowski but have transposes that are not hidden prdd. Note that the simplified Example 6 was obtained by successive rounding of each entry to the nearest even number and then dividing the matrix A by 2, as far as we could.

8 Acknowledgements

The authors would like to thank the referee for insightful suggestions that improved the presentation. This work was done while the second author visited George Mason University in his sabbatical year. He would like to express his gratitude to George Mason University and Toho University.

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