Finite Sets as Complements of Finite Unions of Convex Sets

by
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Suppose $S \subseteq \mathbb{R}^d$ is a set of (finite) cardinality n whose complement can be written as the union of k convex sets. It is perhaps intuitively appealing that when n is large k must also be large. This is true, as is shown here. First the case in which the convex sets must also be open is considered, and in this case a family of examples yields an upper bound, while a simple application of a theorem of Björner and Kalai yields a lower bound. Much cruder estimates are then obtained when the openness restriction is dropped. For a given set S the problem of determining the smallest number of convex sets whose union is $\mathbb{R}^d \setminus S$ is shown to be equivalent to the problem of finding the chromatic number of a certain (infinite) hypergraph $\mathcal{H}_S$. We consider the graph $\mathcal{G}_S$ whose edges are the 2-element edges of $\mathcal{H}_S$, and we show that, when $d = 2$, for any sufficiently large set S, the chromatic number of $\mathcal{G}_S$ will be large, even though there exist arbitrarily large finite sets S for which $\mathcal{G}_S$ does not contain large cliques.

In memory of Vic Klee.

1. Introduction. Given a finite set $S \subseteq \mathbb{R}^d$, we consider the following question: What is the smallest value of k for which there exist k convex sets $C_1, \dots, C_k \subseteq \mathbb{R}^d$ such that $C_1 \cup \ldots \cup C_k = \mathbb{R}^d \setminus S$? We find upper and lower bounds on this number based upon the dimension d and the cardinality $n = |S|$, and we study certain related graphs and hypergraphs.

This problem has apparently not been given explicit treatment until now. If the question is formulated in terms of the complementary set $X = \mathbb{R}^d \setminus S$, then we are asking for the smallest number of convex sets whose union equals X, and this question has been considered, albeit not for sets

X that are complements of finite sets. McKinney [14], Breen [7], Lawrence, Hare, and Kenelly [12], and Perles and Shelah [17] are among many papers motivated by a result involving the notion of "n-convexity" of Valentine [19] and the search for generalizations of that result. See Matoušek and Valtr [15] for recent results and additional references. Also, the problem of finding linear decision trees having few leaf nodes for a given polyhedral set X is closely related to the question of how few relatively open convex sets are required to write X as a union. See Björner, Lovász, and Yao [5]. (We thank an anonymous referee, who noted the relevance of this reference.) Finally, the "art gallery" problems, in which it is desired to write X as a union of a small number of star-shaped sets, are distant relatives of the one at hand. See Bárány and Matoušek [1], for instance.

We now summarize the contents of the paper. In Section 2, the problem is studied in the case in which the convex sets C_i are additionally required to be open. Given an open polyhedron $X \subseteq \mathbb{R}^d$, denote by k the least number of open convex sets C_i ($1 \leq i \leq k$) such that $X = \bigcup C_i$. We present a method to obtain a lower bound on k in terms of the dimension d and the Betti numbers of X, as a simple consequence of a theorem of Björner and Kalai [4]. With this result and a simple example we obtain that, given k open convex sets in $\mathbb{R}^d$ whose union has complement of cardinality n, $(\lfloor \frac{k}{d} \rfloor - 1)^d \leq n \leq \binom{k-1}{d}$.

In Section 3, we return to the case in which the openness restriction is dropped. The Betti bound doesn't apply to the case of general convex sets; and even when S is a singleton, only two convex sets are required to write the complement as a union, while $k = d + 1$ open convex sets are required. Given the finite set $S \subseteq \mathbb{R}^d$, a hypergraph $\mathcal{H}_S$ is introduced, and using Carathéodory's Theorem it is shown that the minimum k of our problem equals the chromatic number of $\mathcal{H}_S$. We study an example, in which S is the set of four vertices of a rectangle.

Theorem 2 yields that, for a set S of cardinality n affinely spanning $\mathbb{R}^d$ whose complement is a union of k convex sets, one has $n \leq (k-1)\binom{k}{\lfloor \frac{k}{2} \rfloor}^{d-1}$. It follows that, for $S \subseteq \mathbb{R}^d$, if $|S|$ is sufficiently large (with d fixed) then the number of sets required is large. In Theorem 3, it is shown that when S is the vertex-set of a simplex, three convex sets suffice.

In section 5, the graphs $\mathcal{G}_S$ whose edge-sets consist of the 2-element edges of $\mathcal{H}_S$ are studied. Analogous graphs have been studied in connection with n-convex sets. See Matoušek and Valtr [15], where they are dubbed "invisibility graphs." See also Pfender [18] for the related notion of "visibility graph." If $S \subseteq \mathbb{R}^2$ has at least nine elements, then $\mathcal{G}_S$ has a clique with four vertices (Theorem 4). There are arbitrarily large (finite) sets $S \subseteq \mathbb{R}^2$ for which $\mathcal{G}_S$ has no large cliques (Theorem 5), and yet the chromatic number of $\mathcal{G}_S$ can be arbitrarily large (Theorem 6).

In the final Section 6, a list of questions and problems is put forth.

2. Complements of unions of open convex sets. Suppose $C_1, \dots, C_k$ are open convex sets in $\mathbb{R}^d$ having finite complement $S = \mathbb{R}^d \setminus (C_1 \cup \cdots \cup C_k)$ and let n denote the cardinality, $n = \left|S\right|$. We obtain an upper bound on n in terms of k and d as a simple corollary of a theorem of Björner and Kalai.

In [4], Björner and Kalai have given a thorough treatment of the question of the relationship between the f-vector and the Betti numbers of a finite simplicial complex. They have characterized the pairs of vectors f, b that can be obtained as the f-vector and "Betti vector" of a simplicial complex. Additionally, they obtained (among other things) the following theorem, which is useful here.

Theorem of Björner and Kalai. *(Theorem 1.3 of [4].) A sequence $b_0, b_1, \dots$ is the sequence of Betti numbers of some simplicial complex having at most k vertices if and only if there is a Sperner family of subsets of $[k-1]$ that has $b_0 - 1$ singleton sets and, for $j \geq 1$, b_j sets of cardinality $j+1$.*

Given a finite family $\mathcal{C}$ of closed (or open) convex sets in $\mathbb{R}^d$, recall that the *nerve* of $\mathcal{C}$ is the simplicial complex on $[k]$ consisting of the index sets of subcollections of $\mathcal{C}$ whose members have nonempty intersection. By versions of the Nerve Theorem (see Section 10 of the survey article, Björner [3]), we see that the such

nerve of $\mathcal{C}$ and the union of its members have the same Betti numbers.

Corollary 1. *Let X be a closed (open) set in $\mathbb{R}^d$, let $\mathcal{C} = \{C_1, \dots, C_k\}$ be a finite collection of closed (open) convex sets in $\mathbb{R}^d$, and suppose that $X = C_1 \cup \cdots \cup C_k$. Let $b_0, b_1, \dots$ be the Betti numbers of X. Then there is a Sperner family of subsets of $[k-1]$ that has $b_0 - 1$ singleton sets and, for $j \geq 1$, b_j sets of cardinality $j+1$.*

Proof. We need only note that the nerve of the family is a simplicial complex having k vertices and Betti numbers $b_0, b_1, \cdots$. □

In the case of the complement of a disjoint union of compact convex sets we get the next statement.

COROLLARY 2. *Let $S \subseteq \mathbb{R}^d$ be a set which is the union of n pairwise disjoint compact convex sets, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a collection of k open convex sets whose union is $\mathbb{R}^d \setminus S$. Then $n \leq \binom{k-1}{d}$.*

Proof. Note that each C_i may be replaced by an open convex set that misses S, so we may assume without loss of generality that the sets C_i are open. In this case, $b_0 - 1 = 0$ and the only nonzero Betti number of $\mathbb{R}^d \setminus S$ other than b_0 is $b_{d-1} = n$; then k must be large enough that there are n subsets of $[k-1]$ having d elements. □

If S is a set of n distinct points in $\mathbb{R}^d$, then the hypothesis is satisfied, so, letting $\eta(k, d)$ denote the largest n such that there exist k open convex sets in $\mathbb{R}^d$ for which the complement of the union is an n-element set, we have that $\eta(k, d) \leq \binom{k-1}{d}$.

Let m be a positive integer. A family of $m-1$ parallel hyperplanes cuts $\mathbb{R}^d$ into m open convex "slabs." We may take d such families, the hyperplanes in each family being parallel to one of the d coordinate hyperplanes. The complement of the union of the resulting collection of $k = dm$ slabs consists of $n = (m-1)^d$ points. It follows that $\eta(k, d) \geq (\frac{k}{d} - 1)^d$, when d divides k.

It is clear from the proofs of the corollaries that in fact the conclusions hold when it is only assumed that the sets C_i and their nonempty intersections are contractible, or when they are merely acyclic. In the case of such families, the bound of Corollary 2 is tight. To see that this is so, consider a simple arrangement of $k \geq d + 1$ hyperplanes in $\mathbb{R}^d$. We need an arrangement for which the union of the bounded cells is homeomorphic to a d-ball; but, according to Dong [9], every simple arrangement has this property. As is well-known, there are $\binom{k-1}{d}$ bounded d-dimensional cells of the arrangement. Upon expanding each hyperplane slightly to an open slab and restricting attention to the bounded portion, we obtain k open convex sets contained in an open d-ball for which the complement of the union consists of $n = \binom{k-1}{d}$ (compact) polytopes. Topologically, this is what we need for Corollary 2. Observe also that we may shrink the polytopes to singleton sets, providing an example in which the complement of the union consists of $\binom{k-1}{d}$ points.

Letting b denote the largest finite n for which there exist k open convex sets in $\mathbb{R}^d$ whose union has complement consisting of n compact convex sets, a analogously when the complement consists of n points, we have that $(\lfloor \frac{k}{d} \rfloor - 1)^d \leq a \leq b \leq \binom{k-1}{d}$. It is possible to show that all of these inequalities are sometimes strict.

Björner, Kalai, and Yao [5] study the problem of determination of linear decision trees for various polyhedral sets X in $\mathbb{R}^d$. It is desired to find such a tree having as few leaves as possible. It is easily seen that the minimum number of relatively open convex sets required to write X as a union is a lower bound on the number of leaves. Two methods are given to find lower bounds on the number of convex sets. One of these involves computations with volumes; the other involves the Euler characteristic χ of X. For certain sets S the volume argument might be of use but it does not yield a bound for our problem in general. The Euler characteristic yields a simple bound (Theorem 3.1 of [5]). Usually the Betti bound of Corollary 1 is tighter, but it can be more difficult to compute the Betti numbers of X than to compute the Euler characteristic. See Björner and Welker [6] for the Betti numbers of the k-equal manifolds.

3. The hypergraphs $\mathcal{H}_S$ and a planar example. Now we proceed to the case in which the convex sets C_i are not required to be open. There is a hypergraph that has relevance to the problem. Given a finite set $S \subseteq \mathbb{R}^d$, the (infinite) hypergraph, here denoted by $\mathcal{H}_S$, has as its vertex set the complement $\mathbb{R}^d \backslash S$, and a subset E of $\mathbb{R}^d \backslash S$ is an edge of $\mathcal{H}_S$ if E is a minimal set such that $S \cap \operatorname{conv}(E) \neq \emptyset$. Each edge of $\mathcal{H}_S$ has cardinality lying between 2 and $d+1$ (inclusive), and is the set of vertices of a simplex. We are interested in determining the possible values for the chromatic number $\mathrm{c}(\mathcal{H}_S)$. Recall that the chromatic number of a hypergraph is the smallest number of colors needed to color the vertices of the hypergraph in such a way that no edge of the hypergraph is monochromatic. This number is easily described geometrically: It is the smallest positive integer k such that $\mathbb{R}^d \setminus S$ can be written as the union of k convex subsets. (The equivalence follows from the well-known theorem of Carathéodory, stated on page 15 of [11].) Theorem 2 gives a lower bound on the number $\mathrm{c}(\mathcal{H}_S)$ which depends only on the dimension d and the cardinality n of S.

The one-dimensional case, for $S \subseteq \mathbb{R} = \mathbb{R}^1$, is easily dispensed with. The n elements of S partition the number line into $n+1$ intervals, and it is clear that in the most efficient coloring each of these intervals gets its own color; so $c(\mathcal{H}_S) = n + 1$.

The problem is less trivial for $d = 2$. The following theorem treats the case of the set of four vertices of a parallelogram.

THEOREM 1. *If S is the set of vertices of a parallelogram then* $c(\mathcal{H}_S) = 4$.

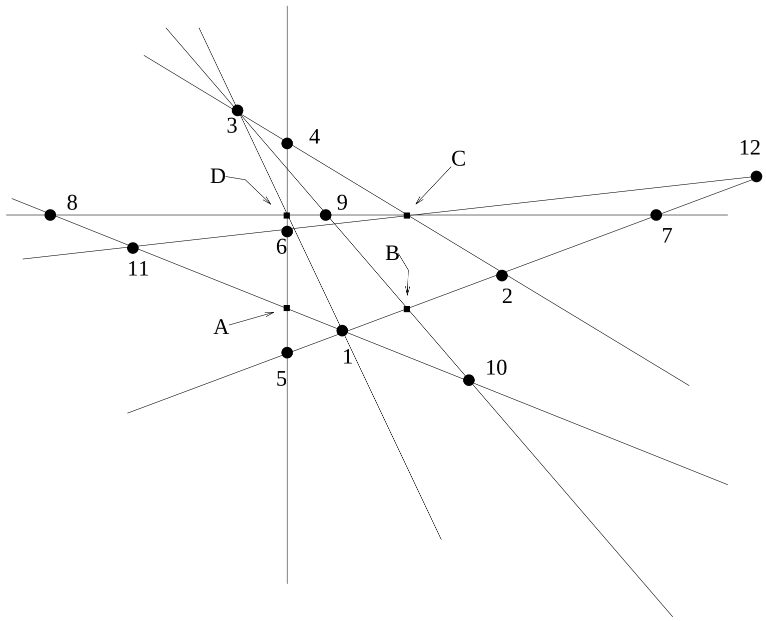

Figure 1. Points in Plane.

Proof. It will be useful to consider, in addition to the hypergraph $\mathcal{H}_S$, the graph $\mathcal{G}_S$ obtained from it by considering only the edges of cardinality two; that is, $\mathcal{G}_S$ is the graph whose vertex set is $\mathbb{R}^d \setminus S$, with two vertices adjacent if the line segment they determine has nonempty intersection with S. It is clear that the chromatic number of the graph is at most that of the hypergraph: $c(\mathcal{G}_S) \leq c(\mathcal{H}_S)$.

We may assume that S is the set $\{A, B, C, D\}$ of vertices of the rectangle depicted in Figure 1, since nonsingular affine mappings of the plane to itself induce isomorphisms of the hypergraphs and graphs. The twelve marked points in that figure induce the subgraph depicted in Figure 2, which is easily seen to have chromatic number 4; so we have $c(\mathcal{H}_S) \geq c(\mathcal{G}_S) \geq 4$. A 4-coloring of $\mathcal{H}_S$ is depicted in Figure 3. □

The chromatic number of $\mathcal{H}_S$ equals the maximum of the chromatic numbers of its finite subhypergraphs; and similarly for $\mathcal{G}_S$ ([12], [8]). Indeed this holds for any graph or any hypergraph having only finite edges, when the chromatic number is finite (assuming the Axiom of Choice).

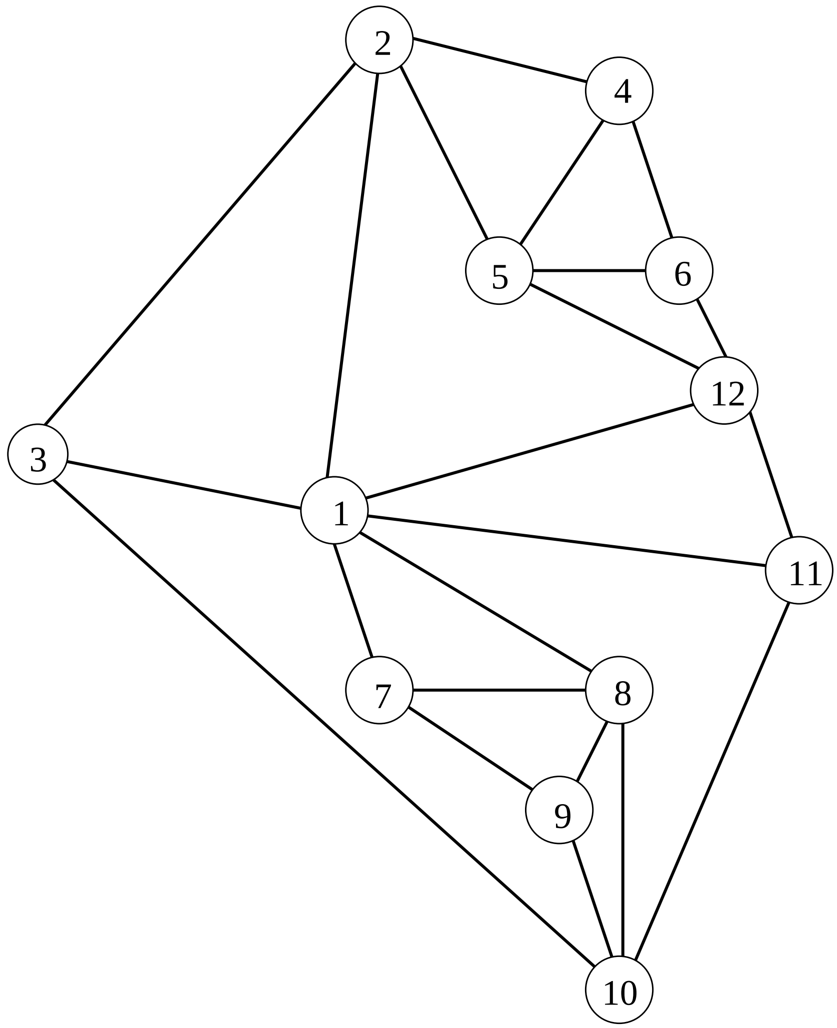

Figure 2. Corresponding Graph.

4. A bound on $\mathbf{c}(\mathcal{H}_S)$. In this section we present a bound on the cardinality of S, given the dimension d and the number k of convex sets in the covering of the complement. We also give a result that shows that no such bound in terms of k alone exists, in contrast to results of Section 2 for the case of open convex sets.

The proof of the next theorem uses the following technical lemma.

Suppose $T = \{p_0, \ldots, p_d\}$ is an affine basis for $\mathbb{R}^d$. We denote by $A(p_0, T)$ the cone emanating from $p_0 \in T$ given by

$$A(p_0, T) = \{p_0 + \sum_{i=1}^{d} \alpha_i (p_0 - p_i) : \alpha_1, \ldots, \alpha_d > 0\}.$$

This is the reflection through p_0 of the interior of the cone emanating from

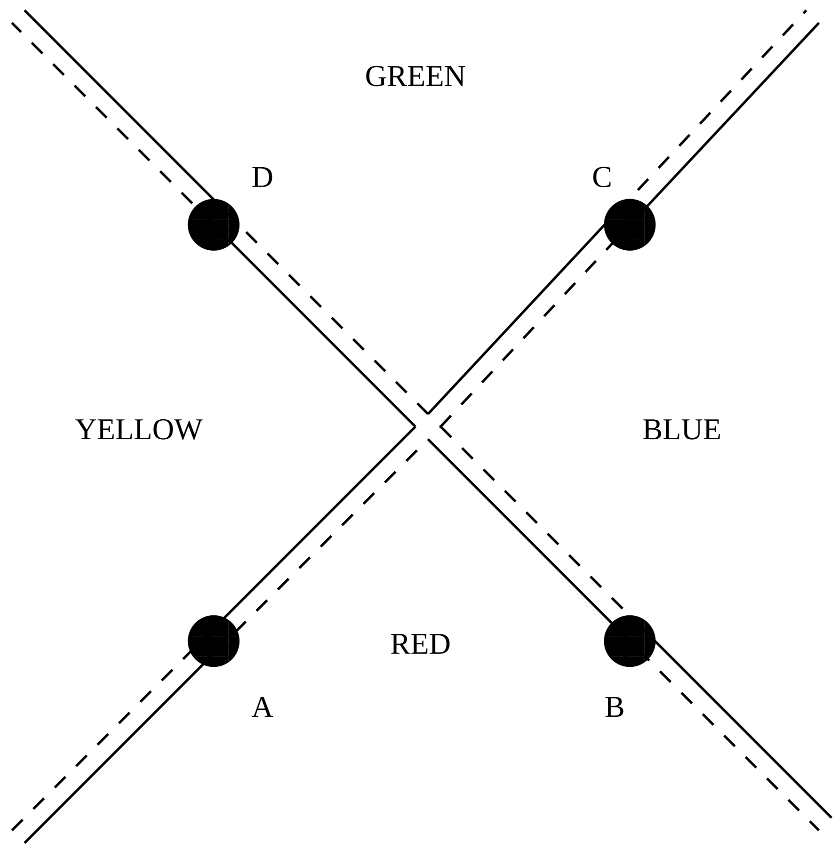

Figure 3. A Four-Coloring.

p_0 generated by T.

LEMMA 1. *Suppose $T = \{p_0, \dots, p_d\}$ is an affine basis for $\mathbb{R}^d$, C is a convex set such that $T \setminus \{p_0\} = \{p_1, \dots, p_d\} \subseteq$ cl C, and $C \cap A(p_0, T)$ is nonempty. Then $p_0 \in C$.*

Proof. Let $x = p_0 + \sum_{i=1}^{d} \alpha_i (p_0 - p_i) \in C$, where $\alpha_i > 0$ $(i = 1, \dots, d)$. Then $p_0 = (x + \alpha_1 p_1 + \cdots + \alpha_d p_d)/(1 + \alpha_1 + \cdots + \alpha_d)$ is in the interior of conv$\{x, p_1, \dots, p_d\}$. Also conv$\{x, p_1, \dots, p_d\} \subseteq$ cl C so p_0 lies in the interior of cl C, which is contained in C. □

Suppose $S \subseteq \mathbb{R}^d$ is finite and $C_1, \dots, C_k$ are convex sets such that $\bigcup C_j = \mathbb{R}^d \setminus S$. For $v \in S$, let $\lambda(v) = \{j : v \in \text{cl } C_j\} \subseteq [k]$. Using Lemma 1, it follows that, for any point $v \in S$, the set $\{u \in S : \lambda(u) \supseteq \lambda(v)\}$ is contained in a hyperplane.

THEOREM 2. *Let S be a set of n points affinely spanning $\mathbb{R}^d$, and let $C_1, \dots, C_k \subseteq \mathbb{R}^d$ be convex sets such that $C_1 \cup \cdots \cup C_k = \mathbb{R}^d \setminus S$. Then*

$n \leq (k-1)\binom{k}{\lfloor \frac{k}{2} \rfloor}^{d-1}$. *Equivalently, $c(\mathcal{H}_S) \geq k$, where k is the smallest positive integer for which the above inequality holds.*

Proof. We proceed by induction on d. For $d = 1$, clearly, $n \leq k-1$, as required. Suppose $d > 1$ and that the result holds for $d-1$.

By Lemma 1, if $v \in S$ then $\{u \in S : \lambda(u) \supseteq \lambda(v)\}$ is contained in a hyperplane. Let H be such a hyperplane and note that $H \setminus (C_1 \cup \cdots \cup C_k) = H \cap S$. By the inductive assumption, $|H \cap S| \leq (k-1)\binom{k}{\lfloor \frac{k}{2} \rfloor}^{d-2}$.

Let $\mathcal{T}$ be the collection of (distinct) sets of the form $\lambda(v)$, where $v \in S$, which are minimal with respect to set inclusion. We have $|\mathcal{T}| \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$, by Sperner's Theorem; then, since S is the union of the sets $\{v \in S : \lambda(v) \supseteq T\}$ for $T \in \mathcal{T}$, $|S| \leq (k-1)\binom{k}{\lfloor \frac{k}{2} \rfloor}^{d-1}$. □

Theorem 3. *If S is the vertex set of a d-simplex ($d \geq 1$), then $c(\mathcal{G}_S) = c(\mathcal{H}_S) = 3$.*

Proof. Since this property is invariant under affine equivalence, we may assume that $S = \{p_0, \ldots, p_d\}$, where $p_0 = (0, \ldots, 0, 0, 0)$, $p_1 = (0, \ldots, 0, 0, 1)$, $p_2 = (0, \ldots, 0, 1, 1)$, $\ldots$, and $p_d = (1, \ldots, 1, 1, 1)$.

Given $u = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $u^* = (1 - x_d, 1 - x_{d-1}, \ldots, 1 - x_1)$. Let $A_0 = \{u : u$ and u^* are lexicographically positive$\}$. Let $B = \{u : u$ is lexicographically positive and u^* is lexicographically negative$\}$. Let $C = \{u : u$ is lexicographically negative$\}$. It is clear that $A_0 \cup B \cup C = \mathbb{R}^d \setminus \{p_0, p_d\}$, $B \cap S = C \cap S = \emptyset$, and $A_0 \cap S = \{p_1, p_2, \ldots, p_{d-1}\}$. Let $A = A_0 \setminus \{p_1, p_2, \ldots, p_{d-1}\}$. Then $A \cup B \cup C = \mathbb{R}^d \setminus \{p_0, p_1, \ldots, p_d\}$.

Also it is clear that A_0, B, and C are convex. We show that A is also convex. Let $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ be elements of A and suppose there is a convex combination $w = \alpha u + \beta v$ (so that $\alpha, \beta \geq 0, \alpha + \beta = 1$) which is not in A. Since $A_0 \supseteq A$ is convex, w must be in $A_0 \setminus A$. If j is the smallest index for which one of u_j, v_j is nonzero, then, using lexicographic positivity of u and v, $w_j = \alpha u_j + \beta v_j > 0$. Letting j' denote the largest index for which neither of u_j, v_j is 1, we see by lexicographic positivity of u^* and v^* that $w_{j'} = \alpha u_{j'} + \beta v_{j'} < 1$. Then, since $w \in \{p_1, \ldots, p_{d-1}\}$, $j' < j$. We have that $u_i = v_i = 0$ for $i < j$ and $u_i = v_i = 1$ for $i > j'$, so it must be the case that $j = j' + 1$ and $u = v$ is in $A_0 \setminus A$, a contradiction.

Then A, B, and C are convex and have union $\mathbb{R}^d$. We have $c(\mathcal{G}_S) \leq c(\mathcal{H}_S) \leq 3$.

That $c(\mathcal{G}_S) \geq 3$ is easy to verify (and holds for any set S such that $|S| \geq 2$). □

5. The graphs $\mathcal{G}_S$. One might venture the conjecture that a statement analogous to Theorem 2 should hold for $\mathcal{G}_S$, as well, so that for large sets S the chromatic number of $\mathcal{G}_S$ should be large. One way to attempt to prove this would be to try to find a lower bound on the maximum clique size of $\mathcal{G}_S$ that grows without bound as $|S|$ does. The following theorem shows that such

the graph does contain cliques of four vertices when $|S|$ is large enough. However, there are arbitrarily large (finite) sets S for which $\mathcal{G}_S$ contains no clique having 22 vertices; this is Theorem 5. Subsequently, the last theorem establishes the conjecture when $d = 2$, by making use of a different sequence of graphs having members of large chromatic number.

LEMMA 2. *Assume the vertices of a convex pentagon are y_1, y_2, y_3, y_4, y_5, in clockwise order around the pentagon. There exists a y_j for which the rays $\overrightarrow{y_j y_{j-1}}$ and $\overrightarrow{y_j y_{j+1}}$ both intersect the line L determined by y_{j-2} and y_{j+2}. (Indices are taken modulo 5.)*

Proof. At least one of the rays $\overrightarrow{y_1 y_5}$ and $\overrightarrow{y_1 y_2}$ intersects the line $L_{3,4}$ (where $L_{i,j}$ denotes the line through y_i and y_j). If both do, then y_1 serves as the y_j required by the lemma. Otherwise, assume that $\overrightarrow{y_1 y_5}$ intersects $L_{3,4}$ and that $\overrightarrow{y_2 y_1}$ either intersects or is parallel to $L_{3,4}$. Then $\overrightarrow{y_2 y_1}$ must intersect the line $L_{4,5}$. If also $\overrightarrow{y_2 y_3}$ intersects $L_{4,5}$, then y_2 satisfies the requirements of the lemma. Otherwise, $\overrightarrow{y_3 y_2}$ intersects or is parallel to $L_{4,5}$. Then $\overrightarrow{y_3 y_2}$ intersects $L_{1,5}$. We assumed that $\overrightarrow{y_1 y_5}$ intersects $L_{3,4}$; the point of intersection must be on the ray $\overrightarrow{y_3 y_4}$. Therefore y_3 satisfies the requirements of the lemma. □

THEOREM 4. *Suppose S has nine points. Then $\mathcal{G}_S$ has a clique with four vertices.*

Proof. If S contains three collinear points, then $\mathcal{G}_S$ clearly contains a clique with four vertices. If S has nine points, no three of which are collinear, then S contains the vertex set of a convex 5-gon. (See Morris and Soltan [16] for a proof due to Bonnice.) Let $y_1, \ldots, y_5$ be as in the lemma, indexed so that the j guaranteed by the lemma is $j = 3$. For $\epsilon > 0$, let $x_1 = y_3 + \epsilon(y_3 - y_1) + \epsilon(y_3 - y_5)$. For small enough ϵ, the three intersection points

x_2, x_3, x_4 of the rays $\overrightarrow{x_1y_2}$, $\overrightarrow{x_1y_3}$, and $\overrightarrow{x_1y_4}$ with the line L through y_1 and y_5 will lie in different components of $L \setminus \{y_1, y_5\}$. The vertices x_1, x_2, x_3, x_4 form a clique of $\mathcal{G}_S$. □

By a *transversal* of a family of line segments is meant a set T which has nonempty intersection with the relative interior of each segment of the family. By a *pair-transversal* of a set X is meant a transversal of the family of all line segments determined by pairs of points of X. Given a finite set X in $\mathbb{R}^2$, $\tau(X)$ denotes the minimum cardinality of a pair-transversal of X.

Lemma 3. *Let X be a set of n points in $\mathbb{R}^2$ that affinely spans $\mathbb{R}^2$. Let m denote the number of points of X that lie in the interior of the convex polygon $P =$ conv(X). Then $\tau(X) \geq 2n - 3 + m$.*

Proof. The polygon P can be triangulated (using geometric triangles) in such a way that the points of X are the vertices of the triangulation. There are exactly $2n - 3 + m$ edges in such a triangulation, and these edges have pairwise disjoint relative interiors. □

Lemma 4. *Let X be the set of vertices of a convex 5-gon. Then $\tau(X) = 8$.*

Proof. Any pair-transversal T of X is of the form $T = T_1 \cup T_2$, where T_1 consists of the points of T lying in the interior of $P =$ conv(X) and T_2 consists of boundary points. Then T_2 has at least five elements, one for each boundary edge; and T_1 is a transversal of the set of five interior edges of P, no three of which have relative interiors with a point in common. It follows that T_2 has cardinality at least $\lceil \frac{5}{2} \rceil = 3$, so $|T| = |T_1| + |T_2| \geq 8$. Clearly eight such points can be chosen. □

Lemma 5. *Suppose X is a finite set in the plane, L is a line determined by points of X, and H^+, H^- are the closed halfplanes bounded by L. Then*

$$\tau(X) \geq \tau(X \cap H^+) + \tau(X \cap H^-) - \tau(X \cap L).$$

Proof. Note that T_1 is a pair-transversal of $X \cap H^+$ if and only if $T_1 \cap L$ is a pair transversal of $X \cap L$ and $T_1 \setminus L$ is a transversal of the relatively open edges determined by $X \cap H^+$ which do not lie in L. If T is a pair-transversal of X then $T_1 = T \cap H^+$, $T_2 = T \cap H^-$, and $T \cap L$ are pair-transversals of $X \cap H^+$, $X \cap H^-$, and $X \cap L$, respectively; and

$$|T| = |T_1 \cup T_2| = |T_1 \setminus L| + |T_2| \geq \big(\tau(X \cap H^+) - \tau(X \cap L)\big) + \tau(X \cap H^-).$$

□

LEMMA 6. *Let $X \subseteq \mathbb{R}^2$ with no four points of X on a line.*
(1) If $|X| = 7$ then $\tau(X) \geq 12$.
(2) If $|X| = 12$ then $\tau(X) \geq 23$.
(3) If $|X| = 22$ then $\tau(X) \geq 45$.

Proof. Let $P =$ conv(X).

Suppose $|X| = 7$. If some point of X is interior to P then Lemma 3 yields the desired conclusion. Otherwise, X contains the vertices of a convex 5-gon, P_0. Let X_0 denote the five vertices. by Lemma 4, $\tau(X_0) \geq 8$. If v is one of the remaining two points of X, there is a line L determined by an edge of P_0 that separates P_0 from v. Using this line L as in Lemma 5, we see that $\tau(X_0 \cup \{v\}) \geq 10$. A similar argument using the last point of X yields $\tau(X) \geq 12$.

Suppose $|X| = 12$. If two or more points of X lie in the interior of P then Lemma 3 yields the result. Otherwise, it is clear that there is a line L, bounding halfplanes H^+ and H^-, $|X \cap L| = 2$, and $|X \cap H^+| = |X \cap H^-| = 7$. Then (1) and Lemma 5 yield (2). A similar argument for $|X| = 22$ yields (3). □

THEOREM 5. *There exist arbitrarily large (finite) sets $S \subseteq \mathbb{R}^2$ such that $\mathcal{G}_S$ contains no clique with twenty-two vertices.*

Proof. Let $\Lambda = \{(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_{45}, \mu_{45})\}$ denote a set of 45 ordered pairs (λ, μ), where $1 \leq \lambda < \mu \leq 22$. Consider the real vector spaces V and W, where V consists of $(\mathbb{R}^2)^{22} \times \mathbb{R}^{45}$, an arbitrary element of which can be denoted by

$$X = (x_1, \ldots, x_{22}; \alpha_1, \alpha_2, \ldots, \alpha_{45}),$$

the x_i's being elements of $\mathbb{R}^2$ and the α_i's being real numbers, and where W is $(\mathbb{R}^2)^{45}$, consisting of points $Y = (y_1, y_2 \ldots, y_{45})$, with the y_i's in $\mathbb{R}^2$. Let $F_\Lambda : V \to W$ be the function that takes X to the point Y, where $y_i = \alpha_i x_{\lambda_i} + (1 - \alpha_i) x_{\mu_i}$. Since $\dim W = 90 > 89 = \dim V$, it is clear that this polynomial function is not surjective: The image of F_Λ is an algebraic set of smaller dimension. Therefore there is a polynomial $P_\Lambda(Y)$, not identically zero, such that $P_\Lambda(Y) = 0$ for $Y \in F_\Lambda(V)$.

Denote by U be the vector space having points of the form $Z = (y_1, y_2, \ldots, y_n)$, where the y_i's are in $\mathbb{R}^2$ (and where, in nontrivial cases,

$n \geq 45$). Choose a point $Z \in U$ with no three of the entries of Z collinear and having the property that, whenever (as above) Λ is an indexed set of ordered pairs, $j_1, \ldots, j_{45}$ are distinct indices between 1 and n, and $Y = (y_{j_1}, \ldots, y_{j_{45}})$ has entries obtained from those of Z, we have that $P_\Lambda(Y) \neq 0$. This is possible, since we have only finitely many polynomial equations. Let $S = \{y_1, y_2, \ldots, y_n\}$.

It remains to be shown that $\mathcal{G}_S$ has no clique with 22 vertices. Suppose, on the contrary, that there is such a clique, with vertices x_λ, $1 \leq \lambda \leq 22$. For each pair (λ, μ) with $1 \leq \lambda < \mu \leq 22$, there are $\alpha \in \mathbb{R}$ (between 0 and 1) and $j \in [n]$ such that $\alpha x_\lambda + (1-\alpha)x_\mu = y_j$. Since no three points of S are collinear and each pair of points x_λ are adjacent in $\mathcal{G}_S$, no four of these points can be collinear, so Lemma 6 applies to yield that the image of the function $(\lambda, \mu) \mapsto j$ has cardinality at least 45. Therefore, we may choose a set Λ of 45 pairs $\{(\lambda_i, \mu_i) : 1 \leq i \leq 45\}$ and corresponding j_i's and α_i's, where the j_i's are distinct. Letting X denote the point $(x_1, \ldots, x_{22}; \alpha_1, \alpha_2, \ldots, \alpha_{45}) \in V$ and Y the point $(y_{j_1}, \ldots, y_{j_{45}}) \in V$, we have $Y \in F_\Lambda(X)$, contrary to our choice of Z. □

For the proof of the next theorem, a sequence of graphs, slight variants of which were apparently first studied by A. Gyárfás (see problem 15, page 360, of [2]), will be useful. For $m \geq 2$ let $\mathcal{D}_m$ denote the graph whose vertices are the ordered pairs (i, j) of integers with $1 \leq i < j \leq m$, having (i_1, j_1) and (i_2, j_2) adjacent provided that $j_2 = i_1$ or $j_1 = i_2$. Given a positive integer k, for sufficiently large m, this graph has chromatic number at least k, a consequence of the next lemma. Given this number m, we show in the proof of Theorem 6 that $\mathcal{G}_S$ contains a subgraph isomorphic to $\mathcal{D}_m$, provided that $|S|$ is large enough.

LEMMA 7. *The chromatic number of $\mathcal{D}_m$ equals $\lceil \log_2 m \rceil$.*

Proof. (This statement is a version of the problem of [2] cited earlier.) Given I, a set of vertices of $\mathcal{D}_m$ with no two adjacent, let $A = \{i \in [m] :$ there is $j \in [m]$ with $(i, j) \in I\}$. Clearly, if $(i, j) \in I$ then $i \in A$ and $j \notin A$. Given a k-coloring of $\mathcal{D}_m$, let $A_1, A_2, \ldots, A_k$ be the subsets of $[m]$ corresponding in this way to the vertices of each color. Let $\epsilon : [m] \to \{0, 1\}^k$ be the function having $\epsilon(i) = (\epsilon_1, \ldots, \epsilon_k)$, where ϵ_l is 1 if $i \in A_l$, and 0 otherwise. For $i, j \in [m]$, where $i < j$, there exists l such that (i, j) is of the l-th color; then $i \in A_l$ and $j \notin A_l$. It follows that ϵ is injective, so $m \leq 2^k$.

A coloring using only $\lceil \log_2 m \rceil$ colors can easily be found. □

The proof of the following theorem also uses the notion of "strongly convex position." We say that a set $\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$ is in *strongly convex position* relative to a given set of coordinates for $\mathbb{R}^2$ if the x-coordinates are in (strictly) increasing order (as indexed), and, for each j with $1 < j < n$, the point p_j is below the line through the points p_{j-1} and p_{j+1}. The next two lemmas facilitate the use of this notion.

Lemma 8. *Given positive integers m, k $(2 \leq k < m)$ there exists a positive integer n such that, for any set of at least n points in the plane with no k on a line, there is a subset having m elements, of which no three are collinear.*

Proof. See [11]. This is a special case of the assertion contained in the remark on page four of that book, where the usefulness of the statement is noted. A proof is given in the notes at the chapter's end. □

Lemma 9. *Given positive integers m and k, there exists n such that for any set $S \subseteq \mathbb{R}^2$ having at least n points, no k collinear, there are a choice of coordinates and a subset $S'' \subseteq S$ of m points, such that, with respect to these coordinates, S'' is in strongly convex position.*

Proof. By a well-known theorem of Erdős and Szekeres (of [10]; see also the survey article, Morris and Soltan [16]), there is a number n_0 such that if S' is a set of n_0 points in the plane, no two elements of S' have equal x-coordinates, and no three elements of S' are collinear, then there is a subset $S'' \subseteq S'$ having m elements such that either S'' or $-S''$ is in strongly convex position.

By Lemma 8, there is n such that any set $S \subseteq \mathbb{R}^2$ having more than n elements, no k on a line, contains a subset S' of n_0 points, no three on a line. By a new choice of coordinates, it can be arranged that the x-coordinates of the elements of S' are distinct. Then, by the first paragraph, S' has a subset S'' of cardinality m such that either S'' or $-S''$ is in strongly convex position with respect to the chosen coordinates; and by another change of coordinates, it can be arranged that S'' is in strongly convex position. □

Theorem 6. *For each positive integer k there exists an integer n_0 such that if $S \subseteq \mathbb{R}^2$ and $|S| \geq n_0$ then $c(\mathcal{G}_S) \geq k$.*

Proof. Let n_1 be a positive integer such that, if $m \geq n_1$, then $\mathcal{D}_m$ has chromatic number at least k.

By Lemma 9, there is a number n_0 such that if S is a set having at least n_0 elements and if S has no $k-1$ collinear points then there is a choice of coordinates for $\mathbb{R}^2$ and a subset $S' \subseteq S$ having $2n_1$ elements such that S' is in strongly convex position with respect to these coordinates.

Let $S \subseteq \mathbb{R}^2$ be a set having cardinality at least n_0. If some subset of $k-1$ points of S is collinear then clearly $c(\mathcal{G}_S) \geq k$. Otherwise, there is a choice of coordinates for $\mathbb{R}^2$ and a subset $S' = \{p_1, \ldots, p_{2n_1}\}$ of S such that, with respect to these coordinates, S' is in strongly convex position.

For $1 \leq i < j \leq n_1$, let $q_{i,j}$ denote the point of intersection of the line through p_{2i-1} and p_{2i} with the line through p_{2j-1} and p_{2j}. If $j_2 = i_1$ or $j_1 = i_2$ then q_{i_1,j_1} and q_{i_2,j_2} are adjacent in $\mathcal{G}_{S'}$. It follows that $\mathcal{D}_{n_1}$ is isomorphic to a subgraph of $\mathcal{G}_S$. □

6. Additional questions. We mention a few open questions and problems.

1. Is it the case that, for each finite set $S \subseteq \mathbb{R}^d$, $c(\mathcal{G}_S) = c(\mathcal{H}_S)$?

2. Compare the problem of covering $\mathbb{R}^d \setminus S$ by open sets that are independent in the graph $\mathcal{G}_S$ (that is, having no two elements adjacent) with the problem of coloring the graph $\mathcal{G}_S$. Already when S consists of a single point in $\mathbb{R}^d$, the number of such open sets required to write $\mathbb{R}^d \setminus S$ as the union is $d+1$, whether or not the sets are also required to be convex.

3. If (when $d \geq 2$) $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ are isomorphic graphs, must S and S' be equivalent by a linear transformation?

4. It is shown by Lawrence and Soltan in [13] that the convex hull of the complement of the union of k convex sets is a convex polyhedron, so that if this set is compact then it is a polytope. How many vertices can such a polytope have?

5. Find $c(\mathcal{H}_S)$ or $c(\mathcal{G}_S)$ in specific cases. What are the values, when S is the set of vertices of a regular n-gon? Does the value change for n-gons that are not regular?

6. We have observed that Corollary 1 applies when the open sets C_i and their intersections are assumed only to be acyclic. Is the resulting bound

tight for compact polyhedra in this case? That is, given a compact polyhedron X having Betti numbers $b_0, b_1, \ldots$ and given an integer k such that there exists a Sperner family of subsets of $[k - b_0]$ consisting of b_j sets of cardinality $j+1$ (for $j = 1, 2, \ldots$), is it always possible to cover X by k acyclic subspaces?

7. Does Theorem 6 hold in $\mathbb{R}^d$ when $d > 2$?

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