Coloring Copoints of a Planar Point Set

Walter Morris
Department of Mathematical Sciences
George Mason University
4400 University Drive,
Fairfax, VA 22030, USA

Abstract

To a set of n points in the plane, one can associate a graph that has less than n^2 vertices and has the property that k-cliques in the graph correspond vertex sets of convex k-gons in the point set. We prove an upper bound of 2^{k-1} on the size of a planar point set for which the graph has chromatic number k, matching the bound conjectured by Szekeres for the clique number. Constructions of Erdős and Szekeres are shown to yield graphs that have very low chromatic number. The constructions are carried out in the context of pseudoline arrangements.

1 Introduction

Let X be a finite set of points in $\mathbb{R}^2$. We will assume that X is in general position, that is, no three points of X are on a line. A subset C of X is called *closed* if $C = K \cap X$ for some convex subset K of $\mathbb{R}^2$. The set $\mathcal{C}$ of closed subsets of X, partially ordered by inclusion, is a lattice. If $x \in X$ and A is a closed subset of X that does not contain x and is inclusion-maximal among all closed subsets of X that do not contain x, then A is called a *copoint* of X *attached* to x.

In the more general context of antimatroids, or convex geometries, copoints have been studied in [1], [2] and [3]. It is shown in these references that every copoint is attached to a unique point of X, and that the copoints are the meet-irreducible elements of the lattice of closed sets. Following [2], we use the notation $M(X)$ to denote the set of copoints of X, partially ordered by inclusion. Edelman and Saks [2] initiated the study of the *convex dimension* of X, which is the smallest number of chains needed to cover $M(X)$. Their investigations applied to convex geometries in general. The paper [4] studied the specific case of convex geometries defined by planar point sets.

It is very easy to find the copoints of a planar point set, and to see that there are less than n^2 of them. Denote by $\alpha(A)$ the unique point to which a copoint A is attached. We will then let $\alpha^{-1}(x)$ denote the set of copoints that are attached to an element x. If a subset A of X is a copoint attached to a point $x \in X$, then the convex hull of A is disjoint from x, because $A \in \mathcal{C}$. There is therefore a line ℓ through x so that A is in one of the open halfspaces determined by ℓ. Furthermore, there can be no other points of X in this open halfspace, by the maximality of A. We can list all of the copoints attached to a point x by rotating a directed line around x. Start with a vertical line through x directed from bottom to top. We may assume, without loss of generality, that no two points of X are on the same vertical line. Call the part of the line above x the *head* and the part of the line below x the *tail*. Rotate the line clockwise around x, noting the order in which the points of $X \backslash x$ are met by the line. If a point y is met by the head of the line, write y, and if y is met by the tail of the line, write $-y$. The sequence of $2|X| - 2$ symbols written as the line makes a complete revolution around x, viewed as a circular sequence, is called the *circular local sequence of* x. At one or more places in the circular local sequence of x there will be an element y followed by an element $-z$. At such a place we can find a copoint. Let m be a line through x of which y and z are on the same side. Let H be the open halfspace defined by m that contains y and z. Then $H \cap X$ is a copoint attached to x.

Central to our investigations are three graphs, each having vertex set $M(X)$.

1. The graph $G(X)$ has an edge between copoints A_1 and A_2 if and only if A_1 and A_2 are incomparable sets. The chromatic number $\chi(G(X))$ is the smallest number of chains into which $M(X)$ can be partitioned. By Dilworth's Theorem, $\chi(G(X))$ is also equal to $\omega(G(X))$, the size of the largest clique in $G(X)$.

2. The graph $\Gamma(X)$ has an edge between copoints A_1 and A_2 if and only if $\alpha(A_1) \in A_2 \cup \alpha(A_2)$ and $\alpha(A_2) \in A_1 \cup \alpha(A_1)$.

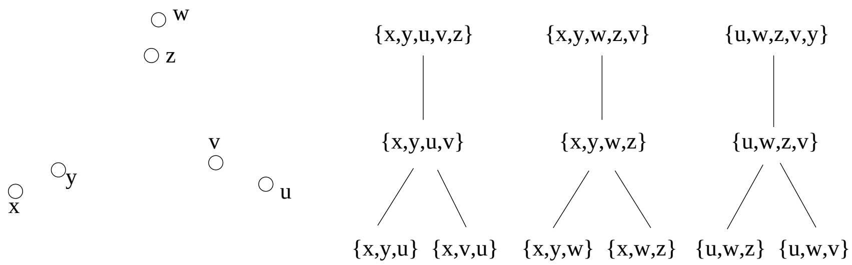

Figure 1: A six point set and its poset of copoints

3. The graph $\mathcal{G}(X)$ has an edge between copoints A_1 and A_2 if and only if $\alpha(A_1) \in A_2$ and $\alpha(A_2) \in A_1$.

Proposition 1.1 *Every edge of $\mathcal{G}(X)$ is an edge of $\Gamma(X)$, and every edge of $\Gamma(X)$ is an edge of $G(X)$.*

Proof. The first part is easy. For each $v \in X$, $\Gamma(X)$ contains the edge set of a clique formed by the copoints A for which $\alpha(A) = v$, and these are the edges of $\Gamma(X)$ that are not in $\mathcal{G}(X)$. To show the second part, note that if $\alpha(A_1) \in A_2$ and $\alpha(A_2) \in A_1$ then A_1 and A_2 are incomparable, because $\alpha(A_1) \notin A_1$ and $\alpha(A_2) \notin A_2$. If $\alpha(A_1) = \alpha(A_2)$ and $A_1 \subseteq A_2$, then $A_1 = A_2$, because A_1 is a maximal convex subset of $X \backslash \alpha(A_1)$. □

If X is the six point set in Figure 1, then the copoints are $\{x, y, u\}$, $\{x, u, v\}$, $\{x, y, w\}$, $\{x, w, z\}$, $\{u, v, w\}$, $\{u, w, z\}$, $\{x, y, u, v\}$, $\{x, y, w, z\}$, $\{u, v, w, z\}$, $\{x, y, u, v, z\}$, $\{x, y, w, z, v\}$, $\{u, v, w, z, y\}$. The copoint $\{x, y, u\}$, for example, is attached to the point v because every closed superset of $\{x, y, u\}$ contains v. The circular local sequence of x is $(w, z, y, v, u, -w, -z, -y, -v, -u)$, and the circular local sequence of y is $(u, -w, -z, x, -v, -u, w, z, -x, v)$. The six three-element copoints induce a clique in $G(X)$, but in $\Gamma(X)$ they only induce a cycle of length six and in $\mathcal{G}(X)$ they induce a graph with three disjoint edges. The set $\{\{x, y, u\}, \{x, u, v\}, \{x, y, w, z, v\}, \{y, u, v, w, z\}\}$ induces a four-clique in $\mathcal{G}(X)$. The chromatic number of $G(X)$ is six, while the chromatic number of $\Gamma(X)$ is four.

Proposition 1.2 *If Y is a k-clique in $\mathcal{G}(X)$, then $\{\alpha(A) : A \in Y\}$ is the vertex set of a convex k-gon in X. If P is the vertex set of a convex k-gon in X, then there is a k-clique Y in $\mathcal{G}(X)$ so that $P = \{\alpha(A) : A \in Y\}$.*

Proof. Suppose Y is a k-clique in $\mathcal{G}(X)$. If $A \in Y$, then $\{\alpha(B) : B \in Y \setminus \{A\}\} \subseteq A$, because Y is a clique. Thus $\alpha(A)$ can be separated from $conv(\{\alpha(B) : B \in Y \setminus \{A\}\})$ by a hyperplane, which means that $\alpha(A)$ is a vertex of $conv(\{\alpha(B) : B \in Y\})$. Next, suppose P is the vertex set of a convex k-gon in X. For each $x \in P$, let $A(x)$ be a copoint attached to x containing $P \setminus \{x\}$. Then $\{A(x) : x \in P\}$ is a k-clique in $\mathcal{G}(X)$. □

The correspondence between k-cliques in $\mathcal{G}(X)$ and convex k-gons in X is not bijective. For a point x in a set P that is the vertex set of a k-gon, there may be several copoints attached to x that contain $P \setminus \{x\}$. A k-gon in X therefore corresponds to a complete k-partite graph in $\mathcal{G}(X)$ that becomes a clique in $\Gamma(X)$. The Proposition shows that the largest number of vertices of a convex polygon in X equals $\omega(\mathcal{G}(X)$, the size of the largest clique in $\mathcal{G}(X)$.

The graphs $G(X), \Gamma(X)$ and $\mathcal{G}(X)$ can be relatively dense, so drawing them directly does not yield much intuition into their structure. In the next section, we show that copoints of a finite point set are much easier to identify and study if one considers the dual setting of line arrangements. In that setting, a copoint corresponds to a face of the arrangement that has exactly one of its bounding lines above it or exactly one of its bounding lines below it.

We summarize known and new results about the parameters ω and χ for the graphs under study.

- Erdős and Szekeres ([5], see also [6]) gave examples of point sets X with $|X| = 2^{k-1}$ and $\omega(\mathcal{G}(X)) = k$. They also conjectured [7] that there are no larger point sets with $\omega(\mathcal{G}(X)) = k$. The currently best upper bound on $|X|$ when $\omega(\mathcal{G}(X)) = k$ is $\binom{2k-3}{k-2}$, due to [8]. See [9] for a survey of results related to this famous problem.

- In [4], specializing results of [10], it was shown that if the chromatic number of $G(X)$ is k, then $|X| \leq 2^{k-1}$.

- We show that a construction of [5] yields sets of $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ points with $\chi(G(X)) = k$. This improves on a construction of B. Aronov and M. Sharir (private communication), who found sets of $2^{\frac{k}{2}}$ points with the same $\chi(G(X))$.

- We prove that $\chi(\mathcal{G}(X)) = k$ implies that $|X| \leq 2^{k-1}$, strengthening the result of [4].

- We show that another construction of [5] yields sets of 2^{k-1} points with $\chi(\Gamma(X)) = k$.

2 Arrangements of (Pseudo-)lines

To a set $X = \{x_1, x_2, \ldots, x_n\}$ of points in the plane, where the coordinates of x_i are (a_i, b_i), for $i = 1, 2, \ldots, n$, we can associate an arrangement L of lines by replacing each point x_i by the line $y = a_i x + b_i$. If we assume that no two points of X lie on the same vertical line, then every pair of lines of L meets at a point. The assumption that no three points of X are collinear implies that no three lines of L meet at a point. This assumption means that the slopes of any two lines, each through two points of X, may be compared.

Recall that for a point $x_i \in X$, we defined the circular local sequence of x_i. Regarding this sequence not as circular but as an ordinary sequence and looking at the first half $(\pm x_{i_1}, \pm x_{i_2}, \ldots, \pm x_{i_{n-1}})$ of the sequence, we note that the corresponding sequence $(\ell_{i_1}, \ell_{i_2}, \ldots \ell_{i_{n-1}})$ gives the order in which line ℓ_i meets the other lines. The appearance of a $+x_{i_j}$ followed by a $-x_{i_{j+1}}$ in this sequence corresponds to line ℓ_i meeting a line ℓ_{i_j} that has a smaller slope than ℓ_i and next meeting a line $\ell_{i_{j+1}}$ that has a greater slope than ℓ_i.

Large sets of almost collinear points present a difficulty when one tries to visualize point sets. The corresponding problem for line arrangements is that some of the vertices (intersection points of lines) may be very close to each other while other vertices are far apart. In order to overcome this difficulty and space the vertices evenly, we will replace the lines of an arrangement by curves that are not straight lines.

A *pseudoline* is the graph of a continuous function from the real line to itself. An arrangement of pseudolines is a set L of pseudolines so that any two of the pseudolines of L have exactly one point in common and cross at their intersection point. We assume that no three lines of the arrangement have a point in common. The *local sequence* of a pseudoline ℓ is the permutation of $L \backslash \ell$ given by the order in which these pseudolines are met by ℓ. A *face* of a pseudoline arrangement L is a connected component of $\mathbb{R}^2 \backslash \cup_{\ell \in L} \ell$. A face F is *above* a pseudoline ℓ if a point of F is directly above a point of ℓ. Otherwise, F is said to be *below* ℓ. The face that is above all of the pseudolines is called the *top face* and the region that is below all of the pseudolines is called the *bottom face*. For every unbounded face F of a pseudoline arrangement L there is a unique unbounded face $\overline{F}$, called the *opposite* of F, which is separated

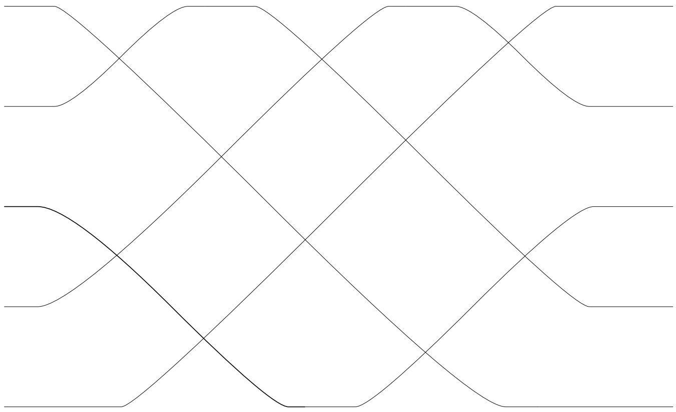

Figure 2: A cycle containing a 4-cap and a 3-cup.

from F by all of the pseudolines of L. A pseudoline arrangement L is said to be *equivalent* to a pseudoline arrangement M if there is a bijection ϕ from L to M so that the local sequence of $\phi(\ell)$ is the image of the local sequence of ℓ, for all $\ell \in L$, and the set of pseudolines of M that touch the top face is the image of the set of pseudolines of L that touch the top face.

Good references for pseudoline arrangements are [11], [12] and [13]. Papers that study pseudoline analogs of the Erdős–Szekeres problem are [14], [15], [16]. A pseudoline arrangement is called *stretchable* if it is equivalent to an arrangement of straight lines. In that case, the pseudoline arrangement is said to be *realized* by the point set that is dual to the arrangement of straight lines.

The example in Figure 4 is a pseudoline arrangement that is realized by the point set of Figure 1.

A pseudoline ℓ in an arrangement L will be said to have *greater slope* than a pseudoline m of L if ℓ is below m for all sufficiently large negative x values.

An arrangement of pseudolines that in which every pseudoline is adjacent to the top face or the bottom face will be called a *cycle*. A cycle consisting of i pseudolines that all touch the top face is called an i-cap, and a cycle consisting of j pseudolines that all touch the bottom face is called a j-cup.

Given two pseudoline arrangements L and M, we can define a *composition* $L \circ M$ to be a pseudoline arrangement of $L' \cup M'$ in which

1. L' is equivalent to L, M' is equivalent to M, and

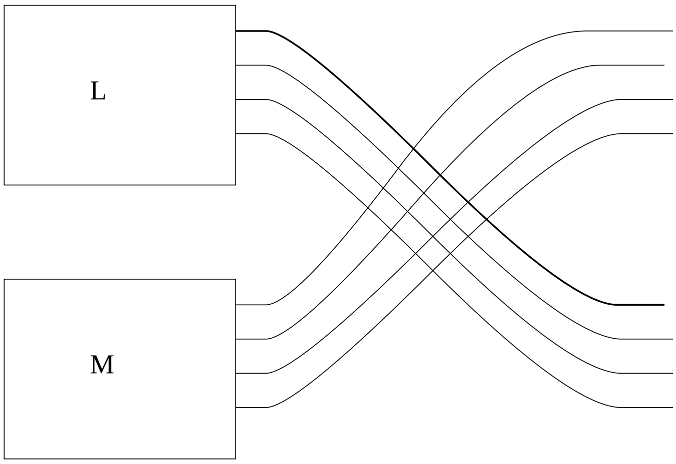

Figure 3: The composition of two pseudoline arrangements

2. for each $\ell \in L'$, the local sequence of ℓ in $L \circ M$ has the pseudolines of M' after the pseudolines of L' and in order of decreasing slope,

3. for each $m \in M'$ the local sequence of m in $L \circ M$ has the pseudolines of L' after the pseudolines of M' and in order of increasing slope, and

4. the pseudolines of L that touch the top face also touch the top face in $L \circ M$.

(See Figure 3.)

The composition operation is neither commutative nor associative, so one can get a large variety of pseudoline arrangements by starting with one-element arrangements and composing them. Many of the pseudoline arrangements to be looked at in this paper, in particular the arrangements $ES(i, j)$ and $XES(k)$ can be obtained this way.

If pseudoline arrangement L is realized by point set X and arrangement M is realized by point set Y, then the composition $L \circ M$ is realizable. To get a realization, transform the point sets X and Y so that they still realize L and M, but so that any line containing two points of X or two points of Y has a small positive slope, any line containing a point of X and a point of Y has negative slope, and the convex hull of X is above the convex hull of Y.

3 Copoints in Pseudoline Arrangements

There exist arrangements of pseudolines that are not realizable by any point sets [11]. We will therefore extend the concept of copoint in the context of pseudoline arrangements. In the case that a pseudoline arrangement is realizable, its copoints, according to the definition for pseudolines, will correspond exactly to the copoints of the realizing set of points.

Suppose that an arrangement of lines L is dual to a point set X. Recall that a copoint $A \subseteq X$ is attached to a point $x \in X$ if there is a line separating A from $X \backslash A$ and the only element of $X \backslash A$ for which the union with A is closed is x. The dual object to the set of lines separating A from $X \backslash A$ is the set of points of a face of the line arrangement L. This motivation is behind the following definitions.

Definition 3.1 *A bounded face F of a pseudoline arrangement L is called a copoint face if there is a pseudoline ℓ so that ℓ is the only line bounding F that is above F or the only line bounding F that is below F. In this case, we say that F is attached to ℓ.*

The definition of an unbounded copoint face is slightly more complicated.

Definition 3.2 *An unbounded face F of a pseudoline arrangement L is a copoint face if there is a pseudoline ℓ so that $\ell = \{$lines bounding $F$and above $F\} \cup \{$lines bounding $\overline{F}$and below $\overline{F}\}$, or $\ell = \{$lines bounding $F$and below $F\} \cup \{$lines bounding $\overline{F}$and above $\overline{F}\}$. Again, we say that F is attached to ℓ in this case.*

A copoint face F attached to a pseudoline ℓ will be called *upward pointing* if ℓ is above F. In that case, the set of lines below F will be the *copoint* corresponding to F. The copoint face F will be called *downward pointing* if ℓ is below F, and then the set of lines above F will be the *copoint* corresponding to F. Note that an unbounded copoint face and the face opposite to it will be attached to the same pseudoline and will correspond to the same copoint.

A sequence of upward pointing copoint faces will be called an *upward chain* if the corresponding sequence of copoints is an increasing sequence of sets. A downward chain is defined similarly.

Figure 4 shows a pseudoline arrangement, with each copoint face labelled by the corresponding copoint.

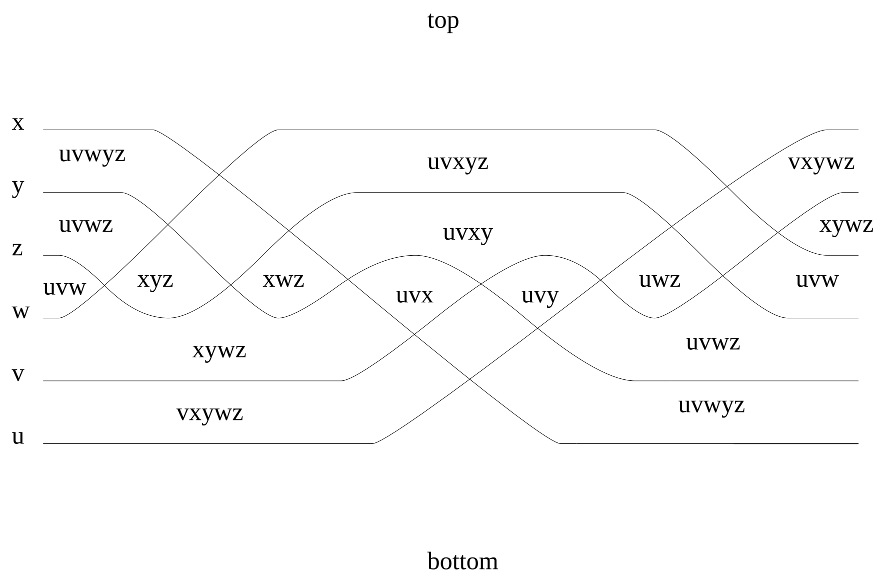

Figure 4: A pseudoline arrangement realized by the points of Figure 1

Now that we have defined copoints for pseudoline arrangement L, we can define the graphs $G(L), \Gamma(L)$ and $\mathcal{G}(L)$ exactly as their counterparts for point sets were defined in the introduction. The analog of Proposition 1.1 is immediate, and it is also clear that, analogous to Proposition 1.2, a clique in $\mathcal{G}(L)$ arises from a cycle in L.

Suppose that F is a bounded upward pointing copoint face, attached to a line ℓ. Let k be the pseudoline for which $k \cap \ell$ is the left endpoint of the segment $\ell \cap F$, and let m be the pseudoline for which $m \cap \ell$ is the right endpoint of $\ell \cap F$. Then k has smaller slope than ℓ and m has larger slope than ℓ. Thus, an upward pointing bounded copoint face attached to ℓ determines a point in the local sequence of ℓ where a pseudoline with smaller slope than ℓ is followed by a pseudoline with greater slope than ℓ. Similarly, a downward pointing bounded copoint face attached to ℓ determines a point in the local sequence of ℓ where a pseudoline with greater slope than ℓ is followed by a pseudoline with smaller slope than ℓ. A pseudoline ℓ is the pseudoline to which an unbounded copoint face and its opposite are attached if and only if the first and last pseudolines in the local sequence for ℓ both have greater slope than ℓ or both have smaller slope than ℓ. From these observations, it is easy to derive the following proposition.

Proposition 3.3 *The copoint faces attached to a given pseudoline alternate between upward and downward pointing copoint faces. The number of copoints attached to a given line is odd.*

We will now translate the construction, due to Erdős and Szekeres [5] of large point sets without large subsets in convex position into the pseudoline framework and show that the resulting pseudoline arrangements have low convex dimension. For each pair of positive integers (i, j), we will define a pseudoline arrangement $ES(i, j)$ that has $\binom{i+j}{i}$ pseudolines. For all positive integers i, j, we first define $ES(1, j)$ to be a $(j + 1)$-cup and $ES(i, 1)$ to be an $(i + 1)$-cap. For $i \geq 2, j \geq 2$ define $ES(i, j) = ES(i - 1, j) \circ ES(i, j - 1)$. Recall that the composition of two pseudoline arrangements was defined in the previous section. Figure 5 shows $ES(2, 3)$, which is the composition of the 4-cup $ES(1, 3)$ and the arrangement $ES(2, 2)$.

Proposition 3.4 *The number of pseudolines of $ES(i, j)$ is $\binom{i+j}{i}$.*

Proof. This is immediate from the initial conditions $|ES(i, 1)| = i + 1$, $|ES(1, j)| = j+1$, and the formula $|ES(i, j)| = |ES(i-1, j)|+|ES(i, j-1)|$. $\square$

The proof of the following proposition contains some redundancy in its description of the unbounded faces on the left and on the right sides of the arrangement. This suggests identifying opposite faces of an arrangement, considering the arrangement in the projective plane. This approach is often taken in the study of pseudoline arrangements (see [12]). The $E(i, j)$ arrangements will be used as blocks in the construction of arrangement $XES(k)$ of the next section, where there is no longer such a symmetry between the left and right sides. It therefore seems necessary to use a fixed left and right side, and describe each side explicitly.

Proposition 3.5 *The set of copoint faces of $ES(i, j)$ can be partitioned into $i + 1$ upward chains and $j + 1$ downward chains in such a way that (1) one of the upward and one of the downward chains consists only of infinite faces on the left side, and (2) one of the upward and one of the downward chains consists only of infinite faces on the right side.*

Proof. We prove this by induction. A cap $ES(i, 1)$ has $i + 3$ copoint faces, one adjacent to every incidence between a pseudoline and either the top or bottom face. There are $i+1$ of these copoint faces under pseudolines touching the top face, and 2 over pseudolines that touch the bottom face. Each of these faces represents a chain of length one, and the condition on chains of infinite faces is clearly satisfied. Similarly, a cup $ES(1,j)$ has $j+1$ copoint faces over the bottom face and 2 copoint faces under the top face. For the inductive step, we will assume that $ES(i-1,j)$ and $ES(i,j-1)$ satisfy the requirements of the proposition. The bounded copoint faces of $ES(i-1,j)$ and $ES(i,j-1)$ are still copoint faces of $ES(i,j) = ES(i-1,j) \circ ES(i,j-1)$. The upward pointing unbounded copoint faces on the right side of $ES(i-1,j)$ and the downward pointing unbounded copoint faces on the right side of $ES(i,j-1)$ become bounded copoint faces of $ES(i,j)$. However, the downward pointing unbounded copoint faces on the right side of $ES(i-1,j)$ and the upward pointing unbounded copoint faces on the right side of $ES(i,j-1)$ are not copoint faces of $ES(i,j)$. Every unbounded face on the right side of $ES(i-1,j)$ has an additional bounding pseudoline between it and the bottom face of $ES(i,j)$, namely the pseudoline of $ES(i,j-1)$ that is adjacent to the top of $ES(i,j-1)$ on the right side. The copoint faces of $ES(i,j)$ that do not correspond to bounded copoint faces of $ES(i-1,j)$ and $ES(i,j-1)$ or unbounded faces on the right side of these arrangements are all infinite faces. The copoint faces of $ES(i,j)$ can now be partitioned into chains as follows. The upward pointing unbounded copoint faces on the left side, the upward pointing unbounded copoint faces on the right side, the downward pointing unbounded copoint faces on the left side, the downward pointing unbounded copoint faces on the right side make two upward and two downward chains. The $i-1$ upward pointing chains of bounded copoint faces of $ES(i,j-1)$ can be matched with the $i-2$ upward pointing chains of bounded copoint faces of $ES(i-1,j)$ and the chain of previously infinite upward pointing copoint faces on the right side of $ES(i,j-1)$ to get $i-1$ more upward chains. Similarly, we get $j-1$ downward pointing chains by matching bounded downward chains of $ES(i-1,j)$ with downward chains of $ES(i,j-1)$. ◻

Proposition 3.6 *The convex dimension of $ES(i,j)$ is $i+j$.*

Proof. The arrangement $ES(i,j)$ has $i+1$ faces adjacent to the top face and $j+1$ faces adjacent to the bottom face. These faces are all copoint faces. They correspond to $i+j$ pairwise incomparable copoints, attached to the $i+j$ lines bounding the top and bottom faces. The convex dimension of $ES(i,j)$

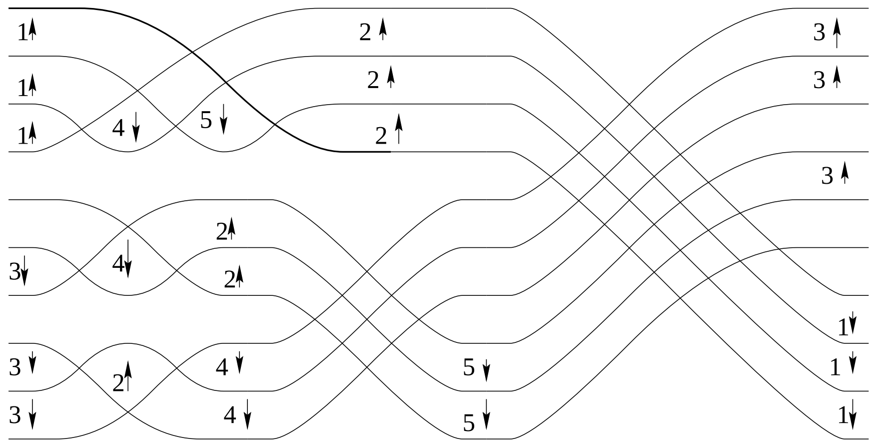

Figure 5: The arrangement $ES(2,3)$.

is therefore at least $i+j$. A chain of upward unbounded copoint faces on the left of $ES(i,j)$ and a chain of downward unbounded copoint faces on the right of $ES(i,j)$ correspond to the same chain of copoints. Similarly, a chain of downward unbounded copoint faces on the right of $ES(i,j)$ and a chain of upward unbounded copoint faces on the left of $ES(i,j)$ correspond to the same chain of copoints. Therefore, the construction above shows that the copoints of $ES(i,j)$ can be partitioned into $i+j$ chains. $\square$

Corollary 3.7 *There exist point sets with* $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ *points that have convex dimension* k*.*

The Corollary shows that the maximum number of lines of an arrangement of convex dimension k can be an exponential function of k. The function given by the Corollary is not the best possible, however. For convex dimension $k = 5$, there is a stretchable arrangement with 15 lines and a nonstretchable arrangement with 16 lines. More generally, for any $k > 4$, [4] gives a nonstretchable arrangement with $\binom{k}{4} + \binom{k}{2} + \binom{k}{0}$ (larger than $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ for small k) lines and convex dimension k. These are shown there to be the largest possible arrangements of convex dimensions $k = 5$ and $k = 6$, leading to that paper's conjecture that the upper bound might be polynomial. The upper bound of 2^{k-1} was proved in [4] and also follows from the results of the next section.

4 Relaxed Copoint Coloring

The graph $\mathcal{G}(L)$ has as its vertices the copoints of L, with copoints A_1 and A_2 adjacent if and only if $\alpha(A_1) \in A_2$ and $\alpha(A_2) \in A_1$.

For a pseudoline arrangement L, and a pseudoline $\ell \in L$, we will define the circular ordering $\mathcal{O}_\ell$ of the copoints attached to ℓ as follows. First list all of the copoints corresponding to upward pointing copoint faces attached to ℓ from left to right. Then append to this list the copoints corresponding to downward pointing copoint faces attached to ℓ, from left to right. If necessary, remove the duplicate copy of a copoint corresponding to an unbounded copoint face, so that each copoint attached to ℓ appears exactly once in the list. Finally, view the resulting list as a circular sequence.

Lemma 4.1 *Assume that there are $2r+1$ copoints attached to a pseudoline ℓ. A set S of $r+1$ copoints is a consecutive set of copoints in $\mathcal{O}_\ell$ if and only if there is a pseudoline m so that S is the set of copoints attached to ℓ that contain m.*

Proof. Suppose that $(A_1, A_2, \ldots, A_{r+1})$ is a contiguous subsequence of $\mathcal{O}_\ell$. Let m be the line crossing ℓ on the left end of the copoint face corresponding to A_1. As Proposition 3.3 shows, the copoint faces met by ℓ as one moves to the right from $\ell \cap m$ alternate between those corresponding to the $r+1$ copoints in $(A_1, A_2, \ldots, A_{r+1})$ and those corresponding to the remaining r copoints, which are $A_{r+2}, A_{r+3}, \ldots, A_{2r+1}$. The last copoint face one encounters before returning to $\ell \cap m$ is A_{r+1}. All of the copoints in $(A_1, A_2, \ldots, A_{r+1})$ contain m. To prove the converse, suppose m is a pseudoline different from ℓ, and let A_1 be the first copoint face attached to ℓ that is met as one moves to the right on ℓ starting from $\ell \cap m$. The contiguous subsequence of $r+1$ copoints of $\mathcal{O}_\ell$ starting with A_1 all must contain m, again by Proposition 3.3. □

Lemma 4.2 *Let A_1, A_2, A_3 be distinct copoints attached to ℓ and let A_2 be in the shorter of the two intervals of $\mathcal{O}_\ell$ that have endpoints A_1 and A_3. If A_1 and A_3 are assigned the same color in a proper coloring of $\mathcal{G}(L)$, then A_2 may be recolored with the same color as A_1 and A_3.*

Proof. If m is a pseudoline distinct from ℓ and m is in A_2, then m is in either A_1 or A_3, by the previous lemma. If ℓ is in a copoint B attached to m, then B must be colored differently from A_1 and A_3 because the coloring of $\mathcal{G}(L)$ is proper. Therefore A_2 may receive the same color as A_1 and A_3. $\square$

From now on, we will assume that any colorings of $\mathcal{G}(L)$ that we use have the property that, for each ℓ, subsequences of $\mathcal{O}_\ell$ receiving the same color are contiguous. We will use the term *odd set* to refer to a set of odd cardinality.

Lemma 4.3 *Suppose $\mathcal{O}$ is a circular sequence of $2r+1$ elements that are colored so that elements receiving the same color are contiguous. Then there is an odd set Ψ of colors so that for every contiguous subsequence of $\mathcal{O}$ of length $r+1$, at least half of the colors in Ψ appear in the sequence.*

Proof. This is clear if $r = 0$ or if $2r+1$ distinct colors are used. If A_1 and A_2 are adjacent elements receiving the same color, let B be the element at distance r from both of A_1 and A_2 in $\mathcal{O}$. Let $\mathcal{O}'$ be the circular sequence obtained from $\mathcal{O}$ by deleting A_1 and B. By induction, there is an odd set Ψ of colors so that each of the contiguous subsequences of length r of $\mathcal{O}'$ is colored by at least half of the colors in Ψ. All but one of the contiguous $(r+1)$-element subsequences of $\mathcal{O}$ contain r-element subsequences of $\mathcal{O}'$. The one that does not is the shorter interval from A_1 to B, but its endpoint A_1 has the same color as A_2, which is in $\mathcal{O}'$, so the shorter sequence between A_1 and B, augmented by A_2 is also colored by at least half of the colors of Ψ. $\square$

For each pseudoline ℓ of L, we will let $\Psi(\ell)$ be an odd set of colors obtained from the previous lemma, with a proper coloring of $\mathcal{G}(L)$.

Lemma 4.4 *If ℓ and m are two different pseudolines of a pseudoline arrangement L, then $\Psi(\ell) \neq \Psi(m)$ for any proper coloring of $\mathcal{G}(L)$.*

Proof. The set of colors of copoints attached to ℓ that contain m must be disjoint from the set of colors of copoints attached to m that contain ℓ. Two disjoint sets cannot both contain at least half of the elements of the same odd set. $\square$

Theorem 4.5 *If L is a pseudoline arrangement and $\chi(\mathcal{G}(L)) = k$, then $|L| \leq 2^{k-1}$.*

Proof. Suppose that L is a pseudoline arrangement and that $\mathcal{G}(L)$ has been properly colored with k colors. Lemma 4.2 shows that we can assume that for every line l, subsequences of $\mathcal{O}_\ell$ receiving the same color are contiguous.

Let Ψ be the function that assigns to each pseudoline an odd set of colors as in Lemma 4.3. Lemma 4.4 shows that Ψ is injective. Therefore its domain cannot have more elements than the number of subsets of odd size of a k-element set, which is 2^{k-1}. $\square$

The analogous but weaker statement for the graph $G(L)$ was proved in [4], partly using arguments from [10]. In [4] it was shown that for $k > 5$ there is no L with $|L| = 2^{k-1}$ and $\chi(G(L)) = k$, and for $k = 5$ the only such L is nonstretchable. In stark contrast are the results of Erdős and Szekeres, who discovered sets of 2^{k-1} points in the plane containing no convex $(k+1)$-gon. In our notation, the corresponding line arrangements L have $|L| = 2^{k-1}$ and $\omega(\mathcal{G}(L)) = k$. We will examine pseudoline arrangements dual to the point sets of Erdős and Szekeres, and show that not only is $\omega(\mathcal{G}(L))$ equal to k, but so is $\chi(\Gamma(L))$ for these arrangements.

Definition 4.6 *For any positive integer k, define $ES(k,0)$ and $ES(0,k)$ both to be an arrangement of one pseudoline. Then the extended Erdős–Szekeres arrangement $XES(k)$ is defined to be $ES(0,k) \circ ES(1,k-1) \circ \cdots \circ ES(k,0)$, where the compositions are performed in order from left to right.*

One can think of the arrangement $XES(k)$ as constructed from a $(k+1)$-cap as follows. If the pseudolines of the cap are numbered $0, 1, \ldots, k$ in order of increasing slope (top to bottom on the left side), we replace line i with a copy of $ES(i, k+1-i)$ on the left end and then continue the pseudolines of $ES(i, k+1-i)$ close to each other and close to the original line i.

The number of pseudolines in $XES(k)$ is $\sum_{i=0}^{k} \binom{k}{i} = 2^k$. It is possible to show that the convex dimension of $XES(k)$ is $2k-2$, for each k. To see that it is at least $2k-2$, note that there are $k-1$ downward pointing copoint faces in the copy of $ES(1, k-1)$ contained in $XES(k)$ and $k-1$ upward pointing copoint faces in the copy of $ES(k-1, 1)$ contained in $XES(k)$. These copoint faces all yield copoints of size $k+1$ and hence must be colored differently in graph $G(XES(k))$. Note that the sets $XES(k)$ are far from the largest arrangements with a given convex dimension.

We will see, however, that the chromatic number of the graph $\Gamma(XES(k))$ is $k+1$. In particular, this means that the copoints of size $k+1$ that needed different colors in graph $G(XES(k))$ do not necessarily do so in graph $\Gamma(XES(k))$. A drawing of $XES(4)$ is given in Figure 6. The 5-coloring of the copoints given is a proper coloring for $\Gamma(XES(4))$. It is not proper for $G(XES(4))$, because there are two copoints close to the left side that are both colored with color 3 and that are adjacent in $G(XES(4))$.

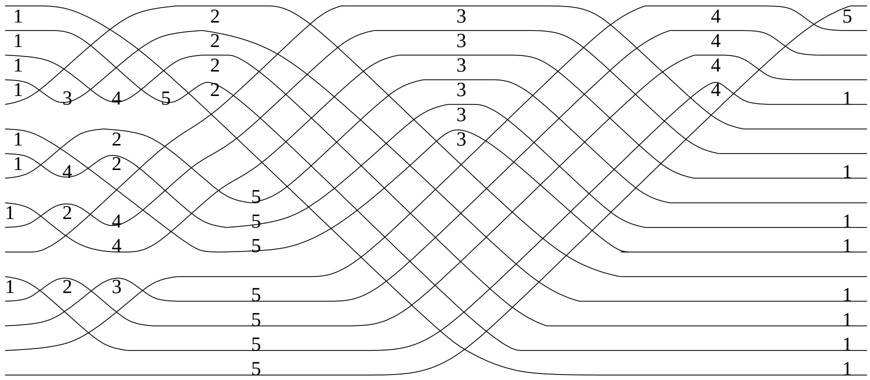

Figure 6: A 5-coloring of XES(4)

Proposition 4.7 $\chi(\Gamma(XES(k))) = k + 1$.

Proof. The reader can refer to Figure 6, in which $k = 4$, to help in visualizing the proof. To see that the chromatic number is at least $k + 1$, note that $XES(k)$ has $k + 1$ lines touching the top face, and the copoints corresponding to faces adjacent to the top face must all be colored with different colors. We will now describe a $(k + 1)$-coloring. The construction of $XES(k)$, in particular the final summand $ES(0, k)$, implies that there is only one unbounded downward pointing copoint face on the left side and one unbounded upward pointing copoint face on the right side. These copoint faces are adjacent to the bottom and top faces. The copoint corresponding to these two unbounded faces will be given color $k + 1$. The copoints corresponding to the upward pointing unbounded faces on the left side and the downward pointing unbounded faces on the right side will be assigned color 1. For each of $i = 2, 3, \dots, k$, there is a chain of upward pointing copoint faces that are attached to the pseudolines of $ES(i-1, k-i+1)$ and are below the parts of those pseudolines that come to the right of their intersections with the pseudolines of $ES(i - 2, k - i + 2)$ and to the left of their intersections with the pseudolines of $ES(i, k - i)$. For each i, color the copoints in the corresponding chain with color i. These chains for colors $i = 2, 3, \dots, k$ will be called *special*. In Figure 6, for example, the six pseudolines of $ES(2, 2)$ bound a chain of upward pointing copoint faces, for which the copoints are all colored with the color 3. These copoint faces are to the right of the intersections of the pseudolines of $ES(1, 3)$ with those of $ES(2, 2)$, and to the left of the intersections of the pseudolines of $ES(2,2)$ with those of $ES(3,1)$.

There are two sets of copoints left to color. One set contains those that correspond to unbounded faces on the right side of the $ES(i,k-i)$ arrangements. As argued earlier, these are all pointed downward and form a chain of bounded copoint faces in the extended arrangement $XES(k)$. The copoints corresponding to these faces can all be given the color $k+1$. Together with the unbounded copoint face already assigned color $k+1$, this creates a single chain of copoints given color $k+1$. (See copoints marked 5 in Figure 6.) Because the colors 1 and $k+1$ will not be used on any of the remaining copoints, we see that each of the colors 1 and $k+1$ is assigned to a chain of copoints, ensuring that no two copoints with one of these colors are adjacent in $\Gamma(XES(k))$.

The remaining set of copoint faces to color is the set of faces that are bounded faces of the $ES(i,k-i)$ arrangements. Recall that within each $ES(i,k-i)$ arrangement, we have $i-1$ chains of bounded upward pointing copoint faces and $k-i-1$ chains of bounded downward pointing copoint faces. For each $i=1,2,\ldots,k-2$, we arbitrarily choose a bijection ϕ_i from the set of colors $\{2,3,\ldots,k-i\}$ to the chains of upward pointing copoint faces of $ES(i,k-i)$. For each $i=1,2,\ldots,k-2$, we arbitrarily choose a bijection ρ_i from the set of colors $\{2+i,2+i+1,\ldots,k\}$ to the chains of downward pointing copoint faces of $ES(i,k-i)$. Each copoint corresponding to a bounded face of $ES(i,k-i)$ receives the color of the chain that contains it. We claim that no two copoints that are adjacent in $\Gamma(XES(k))$ receive the same color. For $i=2,3,\ldots,k-1$ there is a chain of bounded copoint faces of color i in each of $ES(k-1,1), ES(k-2,2),\ldots,ES(i,k-i)$. For $i=2,3,\ldots,k$ there is also the special chain of copoint faces of color i that are attached to pseudolines of the $ES(i-1,k-i+1)$ subarrangement. This chain is above all of the pseudolines from the subarrangements $ES(k-1,1), ES(k-2,2),\ldots,ES(i,k-i)$. We can therefore concatenate all of the chains of copoints corresponding to upward pointing faces of color i into a single chain. Similarly, for color i, $i=3,4,\ldots,k$, there is a chain of downward pointing copoint faces of color i in each of $ES(1,k-1), ES(2,k-2),\ldots,ES(i-2,k-i+2)$. For each i, these can be concatenated to form a single chain. The only way then, for two copoints of color i to be adjacent in $\Gamma(XES(k))$, is for one to point upward and another to point downward. Suppose that the downward pointing copoint is in $ES(l,k-l)$ for some $l\leq i-2$. Then that copoint is contained in the set of pseudolines of $ES(0,k)\cup ES(1,k-1)\cup\cdots\cup ES(l,k-l)$, and is attached to a pseudoline of $ES(l,k-l)$. For the upward pointing copoint, there are two cases. If the copoint is in the special chain for color i, it contains the downward pointing copoint and can therefore not be adjacent to the downward pointing copoint in $\Gamma(XES(k))$. Otherwise, the copoint is in $ES(m, k-m)$ for some $m \geq i$. Then the upward copoint is contained in the set $ES(k,0) \cup ES(k-1,1) \cup \ldots \cup ES(m, k-m)$ and is attached to a pseudoline of $ES(m, k-m)$. However, no pseudoline of $ES(m, k-m)$, for $m \geq i$ can be in $ES(0,k) \cup ES(1,k-1) \cup \cdots \cup ES(l, k-l)$ when $l \leq i-2$. Therefore, the two copoints of color i are not adjacent in $\Gamma(XES(k))$. $\square$

The final part of the above proof is illustrated in Figure 6, where there is a downward pointing copoint face in $ES(1,3)$, colored with color 3. The corresponding copoint is contained in all of the copoints corresponding to the faces in the special chain for color 3. There is one remaining upward pointing copoint face with color 3, in $ES(3,1)$. The line to which this copoint is attached is not in the downward pointing copoint, so these copoints are not adjacent in $\Gamma(XES(4))$.

The results of this and the previous section show that the problem of copoint coloring is closely related to previous research on the Erdős–Szekeres problem. A natural question one might ask is if the equation $\omega(G(L)) = \chi(G(L)$, which follows from Dilworth's Theorem, has an analog in the graphs $\Gamma(L)$ and $\mathcal{G}(L)$. We do not know the answer for $\Gamma(L)$, but the answer is no for $\mathcal{G}(L)$, due to a counterexample given in Figure 7.

For the arrangement L of Figure 7, one can show that $\omega(\mathcal{G}(L)) = 4$, $\chi(\mathcal{G}(L)) = 5$, $\chi(\Gamma(L) = 7$, and $\chi(G(L)) = 8$.

5 Acknowledgments

This work was done while the author was a member of the Mathematical Sciences Research Institute in the Fall of 2003, in the program on Discrete and Computational Geometry. The author wishes to thank George Mason University for a sabbatical during this time. Discussions with Boris Aronov contributed much insight and motivation to the research that went into the paper.

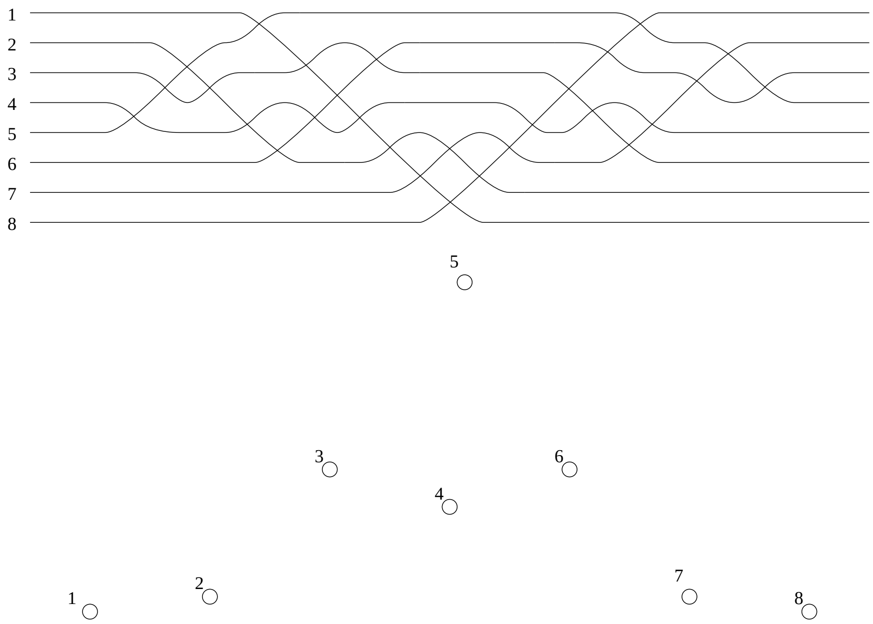

Figure 7: An arrangement of eight lines with no 5-cycle

References

[1] P. H. Edelman and R. E. Jamison, The theory of convex geometries, Geom. Dedicata 19 (1985), 247–270.

[2] P. H. Edelman and M. E. Saks, Combinatorial representation and convex dimension of convex geometries, Order 5 (1988), 23–32.

[3] R. E. Jamison, Copoints in antimatroids, Congressus Numerantium 29 (1980), 535–544.

[4] W. Morris, Convex dimension of locally planar convex geometries, Discr. Comp. Geom. bf 25 (2001), 85–101.

[5] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, Annales Universitatis Scientarium Budapestinensis de Rolando Eötvös Nominatae, Sectio Mathematica, 3 (1960/61) 53–62.

[6] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam - New York, 1979.

[7] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935), 463–470.

[8] G. Tóth and P. Valtr, The Erdős–Szekeres Theorem: upper bounds and related results, in Combinatorial and Computational Geometry (J.E. Goodman, J. Pach and E. Welzl, eds.) MSRI Publications 19 (2005), 557–568.

[9] W. Morris and V. Soltan, The Erdős–Szekeres problem on points in convex position – a survey, Bulletin of the AMS, 37 (2000), 437–458.

[10] S. Hoşten and W. D. Morris Jr., The order dimension of the complete graph, Discrete Mathematics 201 (1999), 133–139.

[11] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, Oriented Matroids, 2nd edition, Cambridge University Press, Cambridge, 1999.

[12] J. E. Goodman, Pseudoline arrangements, in Handbook of Discrete and Computational Geometry, 2nd edition, (J.E. Goodman and J. O'Rourke, eds.), CRC, Boca Raton, (2004), 97–128.

[13] S. Felsner, Geometric Graphs and Arrangements, Vieweg Verlag, 2004.

[14] J. E. Goodman and R. Pollack, A theorem of ordered duality, Geom. Dedicata 12 (1982), 63–74.

[15] J. E. Goodman and R. Pollack, A combinatorial perspective on some problems in geometry, Congressus Numerantium 32 (1981), 383–394.

[16] H. Harborth and M. Möller, The Esther Klein problem in the projective plane, J. Combin. Math. Combin. Comput. 15 (1994), 171–179.