CHROMATIC NUMBERS OF COPOINT GRAPHS OF CONVEX GEOMETRIES

JONATHAN E. BEAGLEY AND WALTER MORRIS

Abstract. We study the copoint graph of a convex geometry. We give a family of copoint graphs for which the ratio of the chromatic number to the clique number can be arbitrarily large. For any natural numbers $1 < d < k$, we study the existence of a number $K_d(k)$ so that the chromatic number of the copoint graph of a convex geometry on a set of at least $K_d(k)$ elements, with every d-element subset closed, has chromatic number at least k. Our results are analogues of results of Erdős and Szekeres for convex geometries realizable by point sets in R^d, where cliques in the copoint graph correspond to subsets of points in convex position.

1. Introduction

Let X be a finite set. An *alignment* $\mathscr{L}$ is a collection of subsets of X such that $\emptyset \in \mathscr{L}, X \in \mathscr{L}$, and if $A, B \in \mathscr{L}$ then $A \cap B \in \mathscr{L}$. A set $C \subseteq X$ is *closed* or *convex* if $C \in \mathscr{L}$. Following Edelman and Jamison [6], we also view $\mathscr{L}$ as a closure operator on the subsets of X, where $\mathscr{L}(A) = \bigcap\{C : C \text{ is closed and } A \subseteq C\}$. The closure operator $\mathscr{L}$ is *anti-exchange* if for any $x, y \notin \mathscr{L}(C), x \in \mathscr{L}(C \cup y)$, then $y \notin \mathscr{L}(C \cup x)$. Equivalently, for any closed set C, with $C \neq X$, there is at least one closed set of the form $C \cup p$ for $p \notin C$. A pair $(X, \mathscr{L})$ where $\mathscr{L}$ is an anti-exchange closure operator is called a *convex geometry*. The closed sets of a convex geometry $(X, \mathscr{L})$ can be partially ordered by inclusion to form a lattice, $L_{\mathscr{L}}$. A subset $A \subseteq X$ is *convexly independent* or *independent* if for all $p \in A, p \notin \mathscr{L}(A - p)$.

A set $C \in \mathscr{L}$ is a *copoint* if it is maximal in $X - p$ for some $p \in X$. If C is a copoint, there is exactly one set in $\mathscr{L}$ of the form $C \cup p$ for $p \notin C$. The unique p is denoted $\alpha(C)$, and we say that the copoint C is *attached* to $\alpha(C)$. We will sometimes refer to a copoint C by the pair $(\alpha(C), C)$. The *copoint graph* of $(X, \mathscr{L})$, $\mathcal{G}(X, \mathscr{L})$, has as its vertex set the set of copoints of $(X, \mathscr{L})$, with copoints C and D adjacent if and only if $\alpha(C) \in D$ and $\alpha(D) \in C$. The definition of independent sets shows that a set $A \subseteq X$ is independent in $(X, \mathscr{L})$, if and only if there is a clique in $\mathcal{G}(X, \mathscr{L})$ of copoints attached to the elements of A. Thus the

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clique number of $\mathcal{G}(X, \mathscr{L})$ equals the size of the largest independent set of $(X, \mathscr{L})$.

If X is a set of points in $\mathbb{R}^d$, and $\mathscr{L} = \{C \subseteq X : X \cap conv(C) = C\}$, then $(X, \mathscr{L})$ is a convex geometry, called the convex geometry *realized* by X. One can show that if the points of a set X are in $\mathbb{R}^d$, then a set $A \subseteq X$ is the vertex set of a convex polytope if and only if A is independent in $(X, \mathscr{L})$. For point sets $X \subseteq \mathbb{R}^2$ in *general position*, that is no three points are on the same line, there is a famous conjecture of Erdős and Szekeres [8] that X contains the vertex set of a convex n-gon whenever $|X| > 2^{n-2}$. Morris [16] proved that for a point set X in general position in $\mathbb{R}^2$, the *chromatic* number of $\mathcal{G}(X, \mathscr{L})$ is at least n whenever $|X| > 2^{n-2}$. This result highlights the need to understand the relationship between the chromatic number and clique number of copoint graphs. We will present several results involving the clique and chromatic number of copoint graphs for general convex geometries. One should keep in mind, however, that convex geometries realized by point sets in $\mathbb{R}^d$ form a small subset of the set of all convex geometries.

In Section 2 we answer a question posed by Beagley [2], giving a family of convex geometries $([n], \mathscr{L}_{d,n})$, for positive integers $d < n$, for which $\omega(\mathcal{G}(X, \mathscr{L})) = d + 1$ and $\chi(\mathcal{G}(X, \mathscr{L})) \geq \lceil \log_2(n + 1) \rceil$. This shows that the chromatic number of $\mathcal{G}(X, \mathscr{L})$ cannot be bounded by a function of the clique number of $\mathcal{G}(X, \mathscr{L})$. The convex geometry $([n], \mathscr{L}_{d,n})$ will have the property that it is *d-free*, i.e. $\mathscr{L}$ will contain every d-element subset of $[n]$. The fact that every d-element subset is closed, together with the alignment property implies that every k-element subset is closed for every integer k with $0 \leq k \leq d$. This is a property that is satisfied by convex geometries realized by point sets in general position in $\mathbb{R}^d$.

In Section 3 we investigate the effect that the d-free property alone will have on the chromatic number of $\mathcal{G}(X, \mathscr{L})$. The first main result we prove is that if $1 < d < k$ there exists a number $K_d(k)$ so that any d-free convex geometry on a set of size at least $K_d(k)$ will have the chromatic number of its copoint graph at least k. We also show that $K_d(d+2) = d+3$ for all $d > 1$, analogous to a result of Esther Klein that every set of 5 planar points in general position contains the vertex set of a convex 4-gon.

To close this introductory section, we give the smallest set of points in the plane in general position for which $\omega(\mathcal{G}(X, \mathscr{L}))$ and $\chi(\mathcal{G}(X, \mathscr{L}))$ differ.

Of the 16 order types of 6 planar points in general position [1], there is only one with this property. It is given in Figure 1. The copoints are shown to the right of the point set, in the

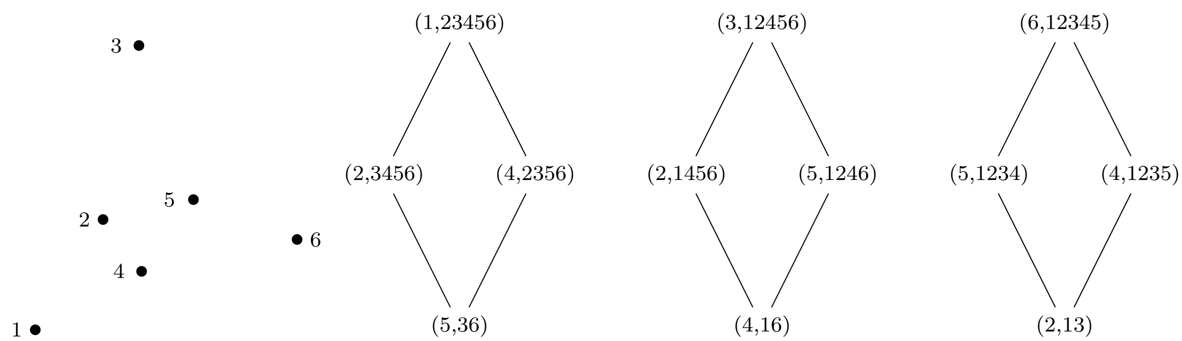

FIGURE 1. A six point set and its poset of copoints

form $(\alpha(C), C)$ where $\alpha(C)$ is the point to which copoint C is attached. The copoints are partially ordered by set containment. The subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the copoints of $(X, \mathscr{L})$ of size bigger than 3 forms the complement of a 9-cycle. This graph has chromatic number 5 and clique number 4. From the figure one can see that the point set does not contain the vertex set of a convex 5-gon, so the clique number of the whole graph is 4.

2. Construction of A Convex Geometry

Beagley [2] asked the following question: Is $\chi(\mathcal{G}(X, \mathscr{L}))/\omega(\mathcal{G}(X, \mathscr{L})) \leq c$ for some constant c? We construct a family of convex geometries indexed by integers d, n with clique number of $d+1$ and chromatic number at least $\lceil \log_2(n+1) \rceil$.

Let n be a positive integer and $\{1, 2, \ldots, n\} = [n]$. When $i = 0$, then $[i] = \emptyset$. Let d be a positive integer, $d < n$, and define $\mathscr{L}_{d,n} = \{([i] \cup J) | 0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq d\}$.

Proposition 2.1. *For n, d positive integers with $d < n$, the pair $([n], \mathscr{L}_{d,n})$ is a d-free convex geometry.*

Proof. It is easy to see that $\mathscr{L}_{d,n}$ is closed under intersection and $\emptyset, [n] \in \mathscr{L}_{d,n}$. Let C be in $\mathscr{L}_{d,n}, C \neq [n]$. If $C = [i] \cup J$ with $0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq d$, then $C \cup \{i+1\} \in \mathscr{L}_{d,n}$, so $([n], \mathscr{L}_{d,n})$ is a convex geometry. To see that $([n], \mathscr{L}_{d,n})$ is d-free, note that if $|J| \leq d$ and i is the smallest element of $[n] \backslash J$, then $J = [i-1] \cup J'$ where $|J'| \leq d$. $\square$

For each $i \in \{1, 2, \ldots, n-d\}$, define $A_i = \{[i-1] \cup J | J \subseteq \{i+1, i+2, \ldots, n\}, |J| = d\}$ and for each $i \in \{n-d+1, n-d+2, \ldots, n\}$ let $A_i = \{[i-1] \cup \{i+1, i+2, \ldots, n\}\}$.

Proposition 2.2. *For $i = 1, 2, \ldots, n$, A_i is the set of copoints of $([n], \mathscr{L}_{d,n})$ attached to i.*

Proof. Suppose $C \in A_i$. Then $C \in \mathscr{L}_{d,n}$ and $i \notin C$. If $j \notin C \cup \{i\}$ then it must be true that $i \in \{1, 2, \ldots, n-d\}$. In that case $|(C \cup \{j\}) \cap \{i+1, i+2, \ldots, n\}| > d$, so $C \cup \{j\} \notin \mathscr{L}_{d,n}$. It follows that C is a copoint attached to i. Suppose $C \in \mathscr{L}_{d,n}$ with $i \notin C$. If $[i-1] \nsubseteq C$ then $C \cup \{j\} \in \mathscr{L}_{d,n}$ for j the smallest element of $[i-1] \backslash C$. Thus C is not a copoint attached to i. If $[i-1] \subseteq C$ and $C \cap \{i+1, i+2, \ldots, n\}$ has fewer than d elements and is not $\{i+1, i+2, \ldots, n\}$, then there exists $j > i$ so that $C \cup \{j\} \in \mathscr{L}_{d,n}$. Again C is not a copoint attached to i. Thus every copoint of $\mathscr{L}_{d,n}$ attached to i is in A_i. □

We define the graph $G_{d,n}$ to be the copoint graph $\mathcal{G}([n], \mathscr{L}_{d,n})$. The size of the maximum clique in $G_{d,n}$ can be found using the size of the largest independent set.

Lemma 2.3. *The clique number of $G_{d,n}$ is $d+1$.*

Proof. Let $C \in \mathscr{L}_{d,n}$. We wish to show that C is the closure in $\mathscr{L}_{d,n}$ of a set of at most $d+1$ elements of $[n]$. If $|C| \leq d$, then $\mathscr{L}_{d,n}(C) = C$. So, let $|C| > d$. We can write $C = [i] \cup J$ where $1 \leq i \leq n-d$, $J \subseteq \{i+1, \ldots, n\}$, $|J| = d$. Thus, $C = \mathscr{L}_{d,n}(\{i\} \cup J)$ and $|\{i\} \cup J| = d+1$. Since every closed set C can be written as the closure of at most $d+1$ elements of $[n]$, there is no independent set of size $d+2$. Thus $\omega(G_{d,n}) \leq d+1$.

Further, $G_{d,n}$ contains a $(d+1)$-clique consisting of the copoints of the form $[n] \backslash \{i\}$ for $i = n-d, \ldots, n$, so $\omega(G_{d,n}) = d+1$. □

Let $f : (V(G_{1,n}) \backslash \{[n-1]\}) \to \binom{[n]}{2}$, where $f([i-1] \cup \{j\}) = \{i, j\}$. Then f is a graph isomorphism from the subgraph of $G_{1,n}$ induced by the vertices other than $[n-1]$ to the shift graph of K_n (see [19, Chapter 8]). The shift graph of K_n is known to have clique number 2 and chromatic number $\lceil \log_2(n) \rceil$.

To bound the chromatic number of $G_{d,n}$ we make note of the following property of the set A_i.

Proposition 2.4. *Suppose that $B \subseteq [n]$, $|B| \leq d$, and $i < b$ for all $b \in B$. Then there exists $C \in A_i$ so that C contains every element of B.*

Proof. Choose a copoint $C = [i-1] \cup J$ in A_i with $B \subseteq J$, and the result is immediate. □

Corollary 2.5. *Suppose that $B \subseteq [n]$, $|B| \leq d$, and that $i < b$ for all $b \in B$. Then there exists $C \in A_i$ so that C is adjacent in $G_{d,n}$ to every copoint D in $\bigcup_{b \in B} A_b$.*

Proof. Proposition 2.4 shows that there is a $C \in A_i$ containing every element of B. Because $b > i$, we have $i \in [b-1] \subseteq D$ for all $D \in A_b$. $\square$

We shall answer Beagley's question, using $([n], \mathscr{L}_{d,n})$ to show that $\chi(\mathcal{G}(X, \mathscr{L}))$ is not bounded by any function of $\omega(\mathcal{G}(X, \mathscr{L}))$.

Theorem 2.6. *The convex geometry* $([n], \mathscr{L}_{d,n})$ *has* $\omega(G_{d,n}) = d + 1$ *and* $\lceil \log_2(n+1) \rceil \leq \chi(G_{d,n})$.

Proof. $\omega(G_{d,n}) = d + 1$ by Lemma 2.3.

For any proper coloring of $G_{d,n}$ with c colors, let S_i be the set of colors used to color the copoints of A_i, $i = 1, 2, \ldots, n$. For $1 \leq i < j \leq n$, the fact that there is a copoint of A_i adjacent to every copoint of A_j means that the S_i are distinct and nonempty. Therefore, $n \leq 2^c - 1$, and any proper coloring of $G_{d,n}$ requires at least $\lceil \log_2(n+1) \rceil$ colors. $\square$

The graph for the convex geometry $([n], \mathscr{L}_{d,n})$ has clique number that is a function of d and independent of n, while the chromatic number is at least $\lceil \log_2(n+1) \rceil$. Therefore the ratio $\chi(G_{d,n})/\omega(G_{d,n})$ can be bigger than any fixed constant c, provided n is large enough.

The precise determination of $\chi(G_{d,n})$ for $d \geq 2$ is an interesting question in its own right. Let S be a finite set. A *d-nondecreasing* sequence of subsets of S is a sequence $S_1, S_2, \ldots, S_t$ so that for any set $B \subseteq [t]$, $|B| \leq d$, and for $j \in [t]$ with $j > b$ for all $b \in B$, we have $S_j \not\subseteq \bigcup_{b \in B} S_b$.

Lemma 2.7. *The chromatic number of* $G_{d,n}$ *is the smallest integer* s *for which there is a* d*-nondecreasing sequence of length* n *of subsets of an* s*-element set* S.

Proof. For any proper coloring of $G_{d,n}$ with s colors, let S_i be the set of colors used to color the copoints of A_i, $i = 1, 2, \ldots, n$. It follows from Corollary 2.5 and the definition of d-nondecreasing sequence of subsets of $[s]$, that $S_n, S_{n-1}, \ldots, S_1$ is a d-nondecreasing sequence of length n.

Let $S_n, S_{n-1}, \ldots, S_1$ be an arbitrary d-nondecreasing sequence of subsets of an s-element set of available colors. It is possible to color the vertices in sets $A_n, A_{n-1}, \ldots, A_1$ successively, because for any $D = [i-1] \cup J$ in A_i that is adjacent to the copoints in A_j for $j \in J$, there is a color in S_i that does not appear in the set S_j for any $j \in J$. This color can be used for the copoint D. Therefore, there is a proper coloring with s colors of $G_{d,n}$. $\square$

We are confronted with the problem of determining the smallest integer s for which there is a d-nondecreasing sequence of length n of subsets of an s-element set S.

A *binary covering array of strength* $d+1$ is an $s \times n$ matrix A with entries in $\{0,1\}$ so that for every $s \times (d+1)$ submatrix B of A, every possible 0-1 vector of length $d+1$ appears as a row of B.

Lemma 2.8. *If A is an $s \times n$ binary covering array of strength $d+1$, then the columns of A are the characteristic vectors of a d-nondecreasing sequence of length n of subsets of an s-element set.*

Proof. If A is an $s \times n$ binary covering array of strength $d+1$ and B is an $s \times (d+1)$ submatrix of A, then there is a row of B which consists of d zeroes followed by a 1. This implies that the set with characteristic vector equal to column $d+1$ of B is not contained in the union of the sets whose characteristic vectors are the first d columns of B. $\square$

The survey paper of Lawrence et al. [14] on covering arrays gives the result of Kleitman and Spencer [13] that there exist C_{d+1} and D_{d+1} such that the smallest integer s for which there exists an $s \times n$ binary covering array of strength $d+1$ is bounded below by $(C_{d+1} - o(1)) \log n$ and above by $(D_{d+1} + o(1)) \log n$.

Corollary 2.9. *There exists a constant D_{d+1} so that the chromatic number of $G_{d,n}$ is at most $(D_{d+1} + o(1)) \log n$.*

2.1. **Order Dimension.** Let $P = (Y, \leq)$ be a partially ordered set. The *order dimension* of P, denoted $\dim(P)$, is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that $P = \cap\mathcal{R}$. Any family of linear extensions $\mathcal{R}$ such that $\cap\mathcal{R} = P$ is called a *realizer* of P.

Felsner and Trotter [9] showed that the order dimension of a partially ordered set P is equal to the chromatic number of a hypergraph for which the vertices are the *critical pairs* of elements of P. Beagley [2] showed that in the case that P is the lattice of closed sets of a convex geometry $(X, \mathscr{L})$, then the the critical pairs are of the form $(\alpha(A), A)$ for copoints A of $(X, \mathscr{L})$. We will describe the hypergraph of critical pairs in this special case.

Let $\vec{\mathcal{G}}(X, \mathscr{L})$ be a directed graph with vertex set equal to the set of pairs $(\alpha(A), A)$ for copoints A of $(X, \mathscr{L})$ and let there be a directed edge from $(\alpha(A), A)$ to $(\alpha(B), B)$

whenever $\alpha(B) \in A$. From this directed graph, we form a hypergraph $\mathcal{H}(X,\mathscr{L})$ on the same vertex set with $\{(\alpha(A_1),A_1),(\alpha(A_2),A_2),\ldots,(\alpha(A_k),A_k)\}$ a hyperedge of $\mathcal{H}(X,\mathscr{L})$ if $\{(\alpha(A_1),A_1),(\alpha(A_2),A_2),\ldots,(\alpha(A_k),A_k)\}$ is a minimal directed cycle in $\vec{\mathcal{G}}(X,\mathscr{L})$. Beagley [2] showed that the digraph $\vec{\mathcal{G}}(X,\mathscr{L})$ is isomorphic to one studied by Felsner and Trotter [9],[19] with the possible addition of vertices that do not appear in any directed cycles. It follows from this correspondence and the results of Felsner and Trotter that $\dim(L_{\mathscr{L}}) = \chi(\mathcal{H}(X,\mathscr{L}))$ [2, Theorem 3.3]. The graph induced by the hyperedges of size 2 in $\mathcal{H}(X,\mathscr{L})$ is $\mathcal{G}(X,\mathscr{L})$. Thus $\dim(X,\mathscr{L}) \geq \chi(\mathcal{G}(X,\mathscr{L}))$ with equality holding whenever $\mathcal{H}(X,\mathscr{L})$ has no hyperedges of size greater than 2.

Proposition 2.10. $\dim(L_{\mathscr{L}_{d,n}}) = \chi(G_{d,n})$

Proof. We show that $\mathcal{H}([n],\mathscr{L}_{d,n}) \cong G_{d,n}$. Suppose that there is a hyperedge of size strictly more than 2, $\{(\alpha(B_1),B_1),(\alpha(B_2),B_2),\ldots,(\alpha(B_k),B_k)\}$ where $k>2$. There is some $i \in [k]$, such that $\alpha(B_i) < \alpha(B_{i+1})$ in $[n]$, so $\alpha(B_i) \in B_{i+1}$, where the indices are taken mod k. Also by definition of $\mathcal{H}([n],\mathscr{L}_{d,n})$, we have that $\alpha(B_{i+1}) \in B_i$. Thus, there is an edge in $G_{d,n}$ between B_{i+1} and B_i. So $\{(\alpha(B_1),B_1),(\alpha(B_2),B_2),\ldots,(\alpha(B_k),B_k)\}$ was not a hyperedge with $k>2$. $\square$

We have shown that the ratio between the chromatic number and the clique number of the graph $G_{d,n}$ can get arbitrarily large. There is a related result about posets in a book of Trotter [19]. The *standard example*, S_n for $n \geq 3$, is a partial order on $X = \{a_1,a_2,\ldots,a_n\} \cup \{b_1,b_2,\ldots,b_n\}$ with the relations $a_i < b_j$ if and only if $i \neq j$, for $i,j = 1,2,\ldots,n$. For $i = 1,2,\ldots,n$, a_i is a minimal element and b_i is a maximal element of the partial order. Figure 2 is the Hasse diagram of the standard example S_5. It is known that the order dimension of S_n is n. However, posets with large order dimension do not require S_n as a subposet. Further, [19] gave examples where the ratio between the order dimension of a poset and the size of the largest subposet isomorphic to a standard example becomes arbitrarily large. Proposition 2.11 shows that the independent sets of $(X,\mathscr{L})$ in $L_{\mathscr{L}}$ act in much the same manner as S_n in posets.

Proposition 2.11. *Let $(X,\mathscr{L})$ be a convex geometry. $L_{\mathscr{L}}$ contains a subposet isomorphic to S_k, the standard example, if and only if $\mathcal{G}(X,\mathscr{L})$ contains a k-clique.*

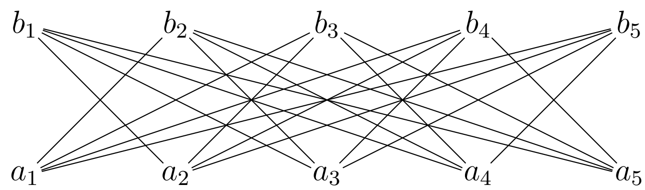

FIGURE 2. The Standard Example, S_5

Proof. We label the standard example S_k contained in $L_{\mathscr{L}}$ in the usual way. Let p_i be a point in X such that $p_i \in (a_i - b_i)$ for $i = 1, 2, \ldots, k$. As $p_i \in a_i$, this means that for all $j \neq i$, $p_i \in b_j$. We now construct the copoint C_i to be a maximal subset of $X - p_i$ containing b_i. Consider the copoints C_i and C_j for $j \neq i$. C_j is a copoint attached to p_j and C_i is a copoint attached to p_i. By definition, $p_i \in b_j \subseteq C_j$ and $p_j \in b_i \subseteq C_i$. This means that C_i and C_j are adjacent in the graph $\mathcal{G}(X, \mathscr{L})$. Since C_i and C_j are adjacent for all $i \neq j$, we have a clique of size k in $\mathcal{G}(X, \mathscr{L})$.

Conversely, let $\mathcal{G}(X, \mathscr{L})$ contain a k-clique composed of copoints $C_1, C_2, \ldots, C_k$ attached to $p_1, p_2, \ldots, p_k$ respectively. By definition of $\mathcal{G}(X, \mathscr{L})$, this means that $p_i \in C_j$ when $i \neq j$. Thus, we let $a_i = \{p_i\}$ and $b_i = C_i$ for $i = 1, 2, \ldots, k$ and we have that $L_{\mathscr{L}}$ contains a subposet isomorphic to S_k. □

Theorem 2.6 and Proposition 2.11 together show that the convex geometry $([n], \mathscr{L}_{1,n})$ and its lattice of closed sets, $L_{\mathscr{L}_{1,n}}$, is an example of a poset that has order dimension that becomes arbitrarily large but does not contain a poset isomorphic to S_3. The lattice $L_{\mathscr{L}_{1,n}}$ is of order dimension k when $|X| = 2^{k-1}$, which means that $|\mathscr{L}_{1,n}| = 2^{2k-3} + 2^{k-2} + 1$. The example given by Trotter [19, Example 5.3] requires a poset of size $R_3(k, 4)$ to have the order dimension equal to k, where $R_3(k, 4)$ is the Ramsey number on 3-regular hypergraphs. It is known that $R_3(k, 4)$ is at least $2^{ck \log(k)}$ for some constant c [3]. Thus the posets $L_{\mathscr{L}_{1,n}}$ perform the function of making the order dimension high at a greater economy than do the examples of [19].

2.2. **Remarks.** Convex geometries isomorphic to $([4], \mathscr{L}_{1,4})$ are in the references [6] and [7]. The copoint graph for $([4], \mathscr{L}_{1,4})$ contains an induced 5-cycle. The convex geometry $([5], \mathscr{L}_{2,5})$, for which the copoint graph has clique number 3, shows that 5 elements do not force a 4-clique for general convex geometries even when every 2-element subset is closed. Thus one would need more restrictions for combinatorial analogues of Esther Klein's result

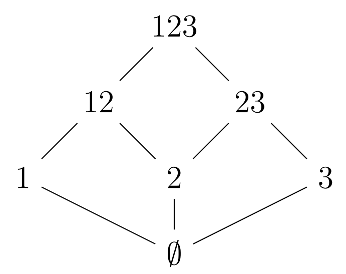

FIGURE 3. Lattice of Closed Sets for a Convex Geometry

that 5 point sets in general position in the plane must contain vertex sets of convex 4-gons. The *chromatic number* of $\mathcal{G}([5], \mathscr{L}_{2,5})$, however, is 4. This will be implied by Theorem 3.6 that we prove in the next section.

One can compute that for any d, n the total number of copoints of the convex geometry $([n], \mathscr{L}_{d,n})$ is $\sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n-d} \binom{n-i}{d} + \sum_{i=n-d+1}^{n} 1 = \binom{n}{d+1} + d$. For the case $d = \lfloor \frac{n-1}{2} \rfloor$, we get that the total number of copoints is $\binom{n}{\lfloor \frac{n}{2} \rfloor} + \lfloor \frac{n-1}{2} \rfloor$. Jamison [11] states that no examples of convex geometries with total number of copoints greater than the middle binomial coefficient for the number of elements are known.

3. Consequences of Freeness

We now introduce a new problem analogous to the Erdős–Szekeres problem: for any integer $k \geq d \geq 2$, determine the smallest positive integer $K_d(k)$ such that for any d-free convex geometry with $|X| \geq K_d(k)$ it follows that $\chi(\mathcal{G}(X, \mathscr{L})) \geq k$. There are two questions of interest related to the study of $K_d(k)$:

1) Does the number $K_d(k)$ exist?

2) If so, how is $K_d(k)$ determined as a function of k?

We specify $d \geq 2$ because of the following 1-free convex geometry. Let $X = [k]$, and for $S \subseteq [k]$ let $\mathscr{L}(S) = [\min(S), \max(S)] \cap X$, which is realizable by a set of k points in $\mathbb{R}^1$. Figure 3 shows this convex geometry for $k = 3$. It is clear that there are two chains of copoints for $(X, \mathscr{L})$, those containing 1 and those containing k. The graph $\mathcal{G}(X, \mathscr{L})$ has chromatic number 2, for all k for this convex geometry, as each of the chains of copoints is an independent set in $\mathcal{G}(X, \mathscr{L})$. This convex geometry has every 1-element subset closed. Thus 1-freeness alone does not force the chromatic number of the copoint graph to increase with $|X|$.

To show that the number $K_d(k)$ exists for $d > 1$, we focus on $K_2(k)$. It is sufficient to show that $K_2(k)$ is finite, because d-freeness for $d > 2$ implies that every 2-element subset is also closed, because $\mathscr{L}$ is an alignment.

Let $(X, \mathscr{L})$ be a 2-free convex geometry and $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$ be a partition of $\mathcal{G}(X, \mathscr{L})$ into independent sets. For $x, y \in X, x \neq y$, define $S_{yx} = \{j \in [t] :$ there is a copoint C with $\alpha(C) = x, y \in C, C \in I_j\}$. For each $x \in X$, let $D_x = \{S_{yx} : y \neq x\}$.

A family of subsets of $[t]$ is called *intersecting* if $A \cap B \neq \emptyset$ whenever A and B are in the family. An intersecting family of subsets is *maximal* if it is contained in no other intersecting family.

Lemma 3.1. *For each $x \in X$, D_x is an intersecting family.*

Proof. A copoint C attached to x is a maximal closed subset in $X - x$. For any $\{y, z\}$ with x, y, and z distinct $\{y, z\}$ is closed, so there is a copoint containing $\{y, z\}$ attached to x. This copoint must be in one of the independent sets I_j. Therefore, $j \in S_{yx} \cap S_{zx}$ and $S_{yx} \cap S_{zx} \neq \emptyset$. □

Corollary 3.2. *No two families D_x for $x \in X$ are contained in the same maximal intersecting family of $[t]$.*

Proof. For $x \neq y$, S_{yx} is contained in the complement of S_{xy} in $[t]$, because $\mathcal{I}$ is a proper coloring of $\mathcal{G}(X, \mathscr{L})$. □

Corollary 3.2 shows that $K_d(k)$ exists and is at most $\gamma(k-1)+1$, where $\gamma(k)$ is the number of maximal intersecting families of a k-element set. The number $\gamma(n)$ of maximal intersecting families of subsets of an n-element set is at most $2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$, see [5] and [12].

The size of the largest antichain of copoints of a convex geometry is called the *convex dimension* of $(X, \mathscr{L})$ [7]. A problem related to computing $K_d(k)$ is that of determining the smallest integer $AC_d(k)$ such that for any d-free convex geometry with $|X| \geq AC_d(k)$ it follows that $(X, \mathscr{L})$ contains an antichain of k copoints. By [7] and [15], the number $AC_d(k)$ is also the answer to *Dushnik's problem*: Find the smallest integer t so that for every set of less than k linear orders of $[t]$ there is a d-element subset J of $[t]$ and some $a \notin J$ such that a is smaller than some element of J in each linear order. Hoşten and Morris [10] and Morris [15] showed that $AC_d(k) \leq \gamma(k-1)+1$ in a manner very similar to the proofs of Lemma 3.1 and Corollary 3.2.

Theorem 3.3. $K_2(k) = \gamma(k-1) + 1$

Proof. Corollary 3.2 shows that $K_2(k) \leq \gamma(k-1) + 1$. The construction of Hoşten and Morris [10] gives a 2-free convex geometry of convex dimension k with $\gamma(k)$ elements, for any k. Edelman and Jamison [6] proved that the convex dimension is bounded below by the order dimension and Beagley [2] proved that the order dimension is bounded below by $\chi(\mathcal{G}(X, \mathscr{L}))$. So, $K_2(k) \geq AC_2(k) \geq \gamma(k-1) + 1$. Therefore, $K_2(k) = \gamma(k-1) + 1$. $\square$

Corollary 3.4. *$K_d(k)$ exists for $d \geq 2$, and $K_d(k+1) \leq 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}} + 1$.*

We do not know of any pair (d, k) for which $AC_d(k) \neq K_d(k)$. Due to the results of [4], $AC_d(k)$ can also be viewed as the smallest positive integer t so that the collection of d-element and 1-element subsets of $[t]$, partially ordered by inclusion, has order dimension k. Known values of $AC_d(k)$ for small d and k are tabulated in [18]. Examples of small convex geometries for which the size of the largest antichain of copoints is larger than the chromatic number of the copoint graph can be found in [7], [16], and Figure 1.

The computation of the numbers $K_d(k)$ for $d > 2$ appears to be difficult, in general. We will calculate $K_d(d+2)$. Before we do this, we recall a result of Morris and Soltan [17] to indicate the kind of combinatorial restrictions that lead to analogous results for the clique number. The *Carathéodory number* of a convex geometry $(X, \mathscr{L})$ is the least positive integer c such that $\mathscr{L}(Y) = \cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq c\}$ for any $Y \subseteq X$.

Let c be the Carathéodory number of a convex geometry $(X, \mathscr{L})$, and suppose that every $(c-1)$-element subset of X is closed. We say that $(X, \mathscr{L})$ satisfies the *simplex partition property* if for any set $\{z_1, z_2, \ldots, z_{c+2}\}$ of $c+2$ elements of X, with $\{z_{c+1}, z_{c+2}\} \subseteq \mathscr{L}(\{z_1, z_2, \ldots, z_c\})$, the point z_{c+2} belongs to exactly one of the sets $\mathscr{L}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_c, z_{c+1})$, $i = 1, \ldots, c$. We state a result of Morris and Soltan [17].

Proposition 3.5. [17, Theorem 5.6] *Let $(X, \mathscr{L})$ be a d-free convex geometry. If $(X, \mathscr{L})$ has Carathéodory number $d+1$, the simplex partition property, and $|X| = d+3$, then X contains $d+2$ convexly independent points.*

The analogous result for chromatic number does not require the simplex partition property or any condition on the Carathéodory number, only that every d-element subset be closed.

Theorem 3.6. *$K_d(d+2) = d+3$ for $d \geq 2$.*

Proof. The example from Section 2, $([d+2], \mathscr{L}_{d,d+2})$, is realizable by a d-simplex with a point in the interior. The copoints of the form $[d+2]\backslash\{i\}$ for $i = 2, 3, \ldots, d+2$ form a $(d+1)$-clique. The remaining copoints are $[d+2]\backslash\{1, i\}$ for $i = 2, 3, \ldots, d+2$. For $i = 2, 3, \ldots, d+2$, the copoint $[d+2]\backslash\{1, i\}$ can be colored with the same color as $[d+2]\backslash\{i\}$, so $\chi(\mathcal{G}([d+2], \mathscr{L}_{d,d+2})) = d+1$. Thus $K_d(d+2) \geq d+3$.

Let $(X, \mathscr{L})$ be a d-free convex geometry with $X = \{q_1, q_2, p_1, \ldots, p_{d+1}\}$. If $(X, \mathscr{L})$ contains a convexly independent set of size $d+2$, we have the conclusion. The elements $x \in X$ for which $X\backslash\{x\} \in (X, \mathscr{L})$ form a convexly independent set. Therefore we assume that there are exactly $d+1$ such elements; specifically, we assume $X\backslash\{p_i\} \in \mathscr{L}$ for $i = 1, 2, \ldots, d+1$. These sets form a clique in $\mathcal{G}(X, \mathscr{L})$.

Let $P = \{p_1, p_2, \ldots, p_{d+1}\}$. For each $i = 1, 2, \ldots, d+1$, the set $P\backslash\{p_i\}$ is closed, because it has cardinality d. Because P is not closed, either $(P\backslash\{p_i\}) \cup q_1$ or $(P\backslash\{p_i\}) \cup q_2$ is in $\mathscr{L}$. Define $I_1 = \{i \in [d+1] : (P\backslash\{p_i\}) \cup q_1 \in \mathscr{L}\}$ and $I_2 = \{i \in [d+1] : (P\backslash\{p_i\}) \cup q_2 \in \mathscr{L}\}$.

Suppose $i \in I_1 \cap I_2$. Let $A = \{(P\backslash\{p_i\}) \cup q_1, (P\backslash\{p_i\}) \cup q_2\} \cup \{X\backslash\{p_j\} : j \neq i\}$. We claim that A is a $(d+2)$-clique in $\mathcal{G}(X, \mathscr{L})$. The copoints $(P\backslash\{p_i\}) \cup q_1$ (attached to q_2) and $(P\backslash\{p_i\}) \cup q_2$ (attached to q_1) are adjacent, and each contains $\{p_j : j \neq i\}$.

Suppose now that I_1 and I_2 are disjoint. Because $d \geq 2$, at least one of these sets must contain two or more elements. Without loss of generality, we assume that $\{1, 2\} \subseteq I_1$. Consider the set $C = (P\backslash\{p_1, p_2\}) \cup \{q_2\}$. $|C| = d$, so C is closed. Because 1 and 2 are not in I_2, the only element of X that can be added to C to form a closed set is q_1. Thus C is a copoint attached to q_1. C is adjacent in $\mathcal{G}(X, \mathscr{L})$ to each $X\backslash\{p_i\}$ for $i \geq 3$, and it is adjacent to each of $(P\backslash\{p_1\}) \cup q_1$ and $(P\backslash\{p_2\}) \cup q_1$. In a proper $(d+1)$-coloring of $\mathcal{G}(X, \mathscr{L})$, the copoints $X\backslash\{p_i\} : i \in [d+1]$ would get distinct colors. $(P\backslash\{p_1\}) \cup q_1$ must get the same color as $X\backslash\{p_1\}$ because it is adjacent to each of $X\backslash\{p_j\}$ for $j > 1$ and $(P\backslash\{p_2\}) \cup q_1$ must get the same color as $X\backslash\{p_2\}$. There is no way to extend such a proper $(d+1)$-coloring to the copoint C. Thus, $\chi(\mathcal{G}(X, \mathscr{L})) \geq d+2$. □

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Department of Mathematical Sciences, George Mason University, 4400 University Drive, Fairfax, VA 22030

Current address, J. Beagley: Department of Mathematics and Computer Science, Valparaiso University, 1900 Chapel Drive, Valparaiso, IN 46383

E-mail address, J. Beagley: `jon.beagley@valpo.edu`

E-mail address, W. Morris: `wmorris@gmu.edu`