ACYCLIC DIGRAPHS GIVING RISE TO COMPLETE INTERSECTIONS

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ABSTRACT. We call a directed acyclic graph a *CI-digraph* if a certain affine semigroup ring defined by it is a complete intersection. We show that if D is a 2-connected CI-digraph with cycle space of dimension at least 2, then it can be decomposed into two subdigraphs, one of which can be taken to have only one cycle, that are CI-digraphs and are glued together on a directed path. If the arcs of the digraph are the covering relations of a poset, this is the converse of a theorem of Boussicault, Feray, Lascoux and Reiner [**2**]. The decomposition result shows that CI-digraphs can be easily recognized.

1. Introduction. The present work applies graph theory to answer a question about partially ordered sets. Suppose $\mathcal{P}$ is a finite partially ordered set (poset) with elements labelled by $\{1, 2, \ldots, m\}$. The function

$$\Psi(\mathcal{P}) = \sum_{w \in \mathcal{L}(P)} \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{m-1}} - x_{w_m})}$$

where $\mathcal{L}(\mathcal{P})$ is the set of linear extensions of $\mathcal{P}$, was introduced by Greene [**10**] and studied by Boussicault and Feray [**1**]. In [**1**] it was shown that this function could be written as a rational function

$$\Psi(\mathcal{P}) = \frac{N(\mathcal{P})}{\Pi_{i \lessdot j}(x_i - x_j)},$$

where $i \lessdot j$ means that i is covered by j. That paper investigated the question: When does the numerator $N(\mathcal{P})$ factor? They showed that if the poset $\mathcal{P}$ can be obtained from smaller posets $\mathcal{P}_1$ and $\mathcal{P}_2$ by *gluing along a chain* (See section 3) then $N(\mathcal{P})$ is the product of $N(\mathcal{P}_1)$ and $N(\mathcal{P}_2)$. The investigations of [**1**] were inspired by a theorem of Greene, [**10**], which showed that $N(\mathcal{P})$ is a product of linear factors if the

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poset $\mathcal{P}$ is *strongly planar* (the Hasse diagram of the poset obtained by appending a maximum and minimum element to $\mathcal{P}$ is a planar graph). The paper left open the problem of determining if a decomposition of $N(\mathcal{P})$ into linear factors implies that $\mathcal{P}$ can be decomposed into smaller posets $\mathcal{P}_1$ and $\mathcal{P}_2$ which can be glued along a chain to form $\mathcal{P}$.

The paper **[2]** showed that $N(\mathcal{P})$ factors into a product of linear factors if the semigroup ring $\mathbb{Z}[S_{\mathcal{P}}]$ is a complete intersection. There it was also shown that if $\mathcal{P}$ is obtained from smaller posets $\mathcal{P}_1$ and $\mathcal{P}_2$ by gluing along a chain, and if $\mathbb{Z}[S_{\mathcal{P}_1}]$ and $\mathbb{Z}[S_{\mathcal{P}_2}]$ are complete intersections, then so is $\mathbb{Z}[S_{\mathcal{P}}]$. This in particular showed that for strongly planar $\mathcal{P}$, the ring $\mathbb{Z}[S_{\mathcal{P}}]$ is a complete intersection.

Recent papers by Gitler et. al. **[8]**, **[9]** have studied complete intersection affine semigroups defined by directed graphs. We will therefore expand the scope of our investigation slightly from Hasse diagrams to directed acyclic graphs. A *directed graph* (digraph) $D = (V, \mathcal{A})$ consists of finite sets V and $\mathcal{A}$ and a mapping associating to each $a \in \mathcal{A}$ an ordered pair of elements of V. A *directed trail* in D is a sequence $(v_0, a_1, v_1, a_2, \ldots, a_k, v_k)$, where $\{v_0, v_1, \ldots, v_k\} \subseteq V$ and $a_1, a_2, \ldots, a_k$ are distinct elements of $\mathcal{A}$ such that for $i = 1, 2, \ldots, k$ the image of a_i is (v_{i-1}, v_i). The digraph D is called *acyclic* if there is no directed trail $(v_0, a_1, v_1, a_2, \ldots, a_k, v_k)$ with $k > 0$ in which v_0 equals v_k. All of the digraphs considered in this paper will be acyclic. Acyclicity implies that every arc will be mapped to an ordered pair of distinct vertices, but it does not exclude parallel arcs, i.e. two arcs mapped to the same ordered pair of vertices. An arc $a \in \mathcal{A}$ is called *dependent* if reversing the ordered pair to which it is mapped makes D no longer acyclic. A directed acyclic graph with no dependent arcs is the Hasse diagram of a partially ordered set for which the arcs are the covering relations of the poset. If a directed acyclic graph has no dependent arcs, we are free to refer to the arcs by the ordered pairs of vertices to which they are mapped.

One can turn an arbitrary directed acyclic graph into a Hasse diagram in two ways. First, one can simply remove dependent arcs one by one until the digraph has no more dependent arcs. The Hasse diagram thus obtained is the Hasse diagram of the transitive closure of the original digraph, but it loses much of the directed graph structure. In particular, the dimension of the cycle space decreases by one for each dependent arc removed. The other way is to replace each dependent

a by a directed path (a_1, a_2) where the tail of a_1 is the tail of a and the head of a_2 is the head of a, and the head of a_1 and tail of a_2 are equal to the same new vertex. We will see in Corollary 4.4 and Lemma 3.2 that both Hasse diagrams obtained have the complete intersection property if and only if the original acyclic digraph does.

Example 0. We start with two vertices, labelled 1 and 6, and four arcs, a, b, c, d, each mapped to the ordered pair $(1, 6)$. This digraph is not the Hasse diagram of a poset. Each of the four arcs is dependent, because reversing any one of them creates a directed path from 1 to 6 and back again along the reversed arc. We could delete three of the arcs to get a Hasse diagram with one arc. If we insert a new vertex into each of the arcs a, b, c, d, we get the Hasse diagram in Figure 1.

The acyclicity condition means that directed trails are directed paths. If the digraph has no dependent arcs, then a directed path is also called a *chain.* A directed path $P = (v_0, a_1, v_1, a_2, \ldots, a_k, v_k)$ in $D = (V, \mathcal{A})$ is called *separating* if there are subdigraphs $D_1 = (V_1, \mathcal{A}_1)$ and $D_2 = (V_2, \mathcal{A}_2)$ of D so that $V = V_1 \cup V_2$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, $V_1 \cap V_2 = \{v_0, v_1, \ldots, v_k\}$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \{a_1, a_2, \ldots, a_k\}$ and such that each of D_1 and D_2 contains an arc not in P. We will say that what is left after removal of a separating path is disconnected, keeping in mind that some of its "components" might be arcs without endpoints. Such arcs would all be dependent arcs of D.

A directed graph is called *connected* if the underlying undirected graph is connected. A digraph is *2-connected* if the removal of any vertex leaves a connected digraph. A 2-connected component of a directed graph is a maximal subdigraph that is 2-connected. The main point of our paper is to show

Theorem 1.1. *If a poset has a complete intersection semigroup ring, then either each 2-connected component of its Hasse diagram has at most one circuit, or it is obtained from smaller posets, each of which has a complete intersection semigroup ring, by gluing along a chain.*

This answers a question which was posed to the author by Reiner and Csar (personal communication). We will actually show the stronger result that the chain referred to by the theorem may be chosen such that one of the smaller posets glued together has at most one circuit.

Section 2 supplies background definitions for complete intersection affine semigroups. The reader with no background in commutative algebra need only take from this section Theorem 2.2, due to Fischer and Shapiro, which characterizes the complete intersection property in terms of the existence of a certain kind of basis of the space of relations of the semigroup. Section 3 specializes the material of Section 2 to affine semigroups arising from directed graphs. One can take the characterization of Fischer and Shapiro as the definition of the class of digraphs we consider, so that all of our presentation can be made in terms of linear algebra. Some of our results, for example Theorem 4, may be proved differently using commutative algebra techniques. In Section 4, we also prove some facts concerning deletion and contraction of arcs in such digraphs. We are particularly interested in cases where these operations preserve the CI-digraph property. Section 5 proves Theorem 1.1. The theorem shows that CI-digraphs are easily recognizable. Section 5 also shows that the basis for the cycle space required by Fischer and Shapiro's condition must be *weakly fundamental*. Finally, in Section 6 we prove that a cycle basis for a CI-digraph satisfying Fischer and Shapiro's condition need not be totally unimodular.

2. Complete Intersection Affine Semigroups. In this section we review the results of **[4]**, **[5]**, **[6]** and **[7]** on general complete intersection affine semigroups, which we will specialize later. Suppose that T is a set of n nonzero vectors in $\mathbb{Q}^m$. Let S be the semigroup generated by T, i.e. the set of nonnegative integer combinations of elements of T. If S contains no invertible elements and if the span of T over $\mathbb{Q}$ has dimension d, then S is called an *affine* semigroup of dimension d. One may associate to S the space of relations W over the rationals $\mathbb{Q}$. If $\{u_1, u_2, \ldots, u_r\}$ is a set of integral vectors (vectors with all integer entries) that forms a basis for W over the rationals, we will call the $r \times n$ matrix with rows $u_1^T, u_2^T, \ldots, u_r^T$ a matrix of relations for W.

Let $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ be the polynomial ring in the variables $X_1, X_2, \ldots, X_n$ over the integers $\mathbb{Z}$. For any vector $u = (u_1, u_2, \ldots, u_n)^T$ in $\mathbb{Z}^n$, define the vectors u^+ and u^- by $u_i^+ = max\{0, u_i\}$ and

$u_i^- = max\{0, -u_i\}$ for $i = 1, 2, \ldots, n$. Define $X^u = \Pi_{i=1}^n X_i^{u_i}$. The semigroup ring $\mathbb{Z}[S]$ is $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ modulo the ideal of relations $\{X^{u^+} - X^{u^-}\}$ where u is any integral vector in W. If this ideal is generated by $r = n - d$ elements, then the semigroup ring $\mathbb{Z}[S]$ is called a complete intersection. In this case, we will also say that S is a complete intersection.

The special case that interests us is where $D = (V, \mathcal{A})$ is an acyclic directed graph, $m = |V|$, $n = |\mathcal{A}|$ and every vector of T is of the form $e_t - e_h$ for an arc of $\mathcal{A}$. Here $e_t(e_h)$ is the standard basis vector with a 1 (-1) in the component given by the tail (head) of the arc. The acyclicity condition on D guarantees that the semigroup S_D generated by T has no invertible elements.

A vector u in $\mathbb{Z}^n$ is called *mixed* if it has both positive and negative components. An $r \times n$ matrix is called *mixed* if each of its rows is mixed. It is called *dominating* if it does not contain a square mixed submatrix. An $r \times n$ matrix M with integer entries is said to have *content 1* if the gcd of its $r \times r$ minors is 1.

A useful fact about mixed dominating matrices is the following (Corollary 2.8 of [**5**]):

Proposition 2.1. *Let M be a mixed dominating matrix. Then the rows of M are linearly independent.*

The following theorem (Corollary 2.10 of [**5**]) characterizes complete intersection affine semigroup rings in terms of the existence of a matrix of relations with certain properties.

Theorem 2.2. *Let S be a finitely generated subsemigroup of $\mathbb{Z}^m$ that contains no invertible elements. Then S is a complete intersection if and only if there exists a basis of integral vectors of the relation space of S whose coefficient matrix is dominating with content 1.*

The mixed dominating property of a matrix was shown to imply a certain block structure, in Theorem 2.2 of [**6**] (see also [**7**]):

Theorem 2.3. *Let M be a mixed dominating $r \times (r+d)$ matrix with $r > 0$. Then there is a rearrangement of the rows and columns of M so that the resulting matrix has the form*

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \\ \hline a & b \end{array}\right] \tag{1}$$

where A and B are mixed dominating matrices of sizes $t \times (t+d_1)$ and $s \times (s+d_2)$ respectively, with $t \geq 0, s \geq 0$ and $s+t+1 = r$ and $d_1 + d_2 - 1 = d$. Additionally, a and b are $1 \times (t+d_1)$ and $1 \times (s+d_2)$ nonzero, nonmixed matrices respectively of opposite sign.

Call the elements of T corresponding to columns of A *red* and the elements of T corresponding to B *green*. A geometric interpretation of the decomposition (1) is that the span of the red elements of T intersects the span of the green elements of T in a line, and the intersection of S with this line is generated by an element α of both of the subsemigroups corresponding to the red and the green elements of T. (Theorem 3.1 of **[6]**).

3. Affine semigroup rings from directed graphs. We will assume throughout that $D = (V, \mathcal{A})$ is an acyclic digraph. The columns of the node-arc incidence matrix of D form a set T of vectors in $\mathbb{Q}^{|V|}$. Let S_D be the semigroup generated by the vectors $e_t - e_h$ in $\mathbb{Q}^{|V|}$ corresponding to arcs of $\mathcal{A}$. If S_D is a complete intersection, then we will say that D is a CI-digraph.

The cycle space is central to the class of digraphs we consider. An excellent survey of recent research on cycle spaces is **[11]**, from which we take much of our notation. Let $W \subseteq \mathbb{Q}^{|\mathcal{A}|}$ be the space of relations of S_D. Then W is the orthogonal complement of the row space of the node-arc incidence matrix of D, called the cycle space of D. By Theorem 2.2, D is a CI-digraph if and only if there is a matrix M that is mixed dominating with content 1, whose rows form a basis of W.

An element of W is called a *cycle* of D. A cycle is called *simple* if its entries are in $\{1, 0, -1\}$, and a simple cycle is called a *circuit* if it is nonzero and has minimal support among simple cycles.

For each circuit u of W, there is a circuit (a connected subgraph with each vertex of degree 2) C_u in the unoriented multigraph underlying D, containing the edges corresponding to arcs indexing the support of u. For one of the two ways to go around C_u, one encounters the arcs for which the component of u is $+1$ in the forward direction and those for which the component of u is -1 in the backward direction. We will use the word circuit to refer to both the vector u and the corresponding subgraph C_u. A vertex v of a circuit C is called a source of C if no arcs of C enter v and it is called a sink of C if no arcs of C leave it. Define $V^+(C)$ to be the set of sources of A and $V^-(C)$ to be the set of sinks of C. The circuit C defines an element $\sum_{i \in V^+(C)} e_i - \sum_{j \in V^-(C)} e_j$ of S_D which can be written as a sum of elements of T in two different ways.

It is known that every integer vector in W can be written *conformally* as a nonnegative rational combination of circuit vectors, that is, a linear combination in which every nonzero component of each of the summands agrees in sign with the corresponding component of the sum.

Theorem 3.1. *Suppose that D is a CI-digraph and let M be the coefficient matrix of a basis of the relation space that is dominating with content 1. Then the rows of M are circuits.*

Proof. If the relation space is 1-dimensional, then it is spanned by a circuit, and the entries of a circuit are ± 1. Now assume that the relation space is r-dimensional for $r > 1$, and that the matrix of the basis is decomposed as in Theorem 2.3 and has content 1. We can assume by induction that the rows of A and B are circuits. Suppose that the vector $u_r = (a, b)$ is not a circuit. Let x be a circuit that conforms to (a, b). Then x is in W and must be an integral combination $x = a_1 u_1 + a_2 u_2 + \cdots a_r u_r$ of the rows of M. The vector $a_r u_r - x$ either conforms to u_r or to the negative of u_r. The vector $a_r u_r - x$ is not the zero vector, because then u_r would be a circuit. Thus the matrix obtained from M by replacing u_r by $a_r u_r - x$ is still dominating. Because its rows are linearly dependent, we have a contradiction to Proposition 2.1. □

The property that all of the rows of M are circuits means that the rows form what **[11]** calls a cycle basis. The property that M has content 1 then implies that **[11]** calls this basis *integral.* A useful technical fact about cycle bases, which can be found in **[11]**, is that if the rows of an $r \times n$ matrix M are a cycle basis of a digraph, then the determinants of all of the $r \times r$ nonsingular submatrices of M have the same absolute value. Thus the content 1 property is equivalent to the property that all nonsingular $r \times r$ submatrices have determinant ± 1.

Example 1. (See Figure 1, where the arcs are all directed upward.) Suppose the vertices of the digraph D are labeled $1, 2, 3, 4, 5, 6$ and the arcs are $(1, 2), (2, 6)$,$(1, 3), (3, 6)$, $(1, 4), (4, 6), (1, 5), (5, 6)$. A matrix of relations is (with the columns labelled in the bottom row.)

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$
$$\begin{matrix} (1,2) & (2,6) & (1,3) & (3,6) & (1,4) & (4,6) & (1,5) & (5,6) \end{matrix}$$

This matrix has content 1 and decomposes as in matrix (1), so D is a CI-digraph. Note that one could replace the last row by $(1, 1, 1, 1, -1, -1, -1, -1)$ and get another set of rows that spans the cycle space and for which the matrix decomposes as in matrix (1). However, the resulting matrix would have content 2.

We can reorder the rows and columns of this example to obtain an alternate decomposition, also in the form of (1):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$
$$\begin{matrix} (1,3) & (3,6) & (1,4) & (4,6) & (1,2) & (2,6) & (1,5) & (5,6) \end{matrix}$$

Here, the vector a consists of the first two entries of the bottom row and the matrix A is empty, i.e. it has no rows. It will follow from our results (Theorem 5.12) that every mixed dominating matrix of relations with content 1 for a CI-digraph decomposes in this way, with either A or B empty.

If an acyclic digraph D has no dependent arcs, then it is a Hasse diagram of a poset. For such D, Boussicault et. al. **[2]** proved that

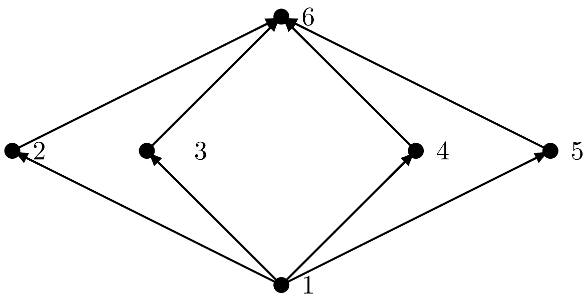

FIGURE 1. Poset for Example 1

D is a CI-digraph if and only if each of the 2-connected components of D is a CI-digraph. It was also proved in [**2**] that if D is a circuit, D is a CI-digraph. These results can also be established from the decomposition in Theorem 2.3 and hold for general directed acyclic graphs. If M is a matrix of relations for S_D, then it can be decomposed into blocks corresponding to the 2-connected components, and M is mixed dominating with content 1 if and only if each of its blocks is. If D has only one circuit, then its matrix of relations has only one row with entries in $\{-1, 0, 1\}$, which makes it mixed dominating with content 1.

Lemma 3.2. *Suppose that acyclic digraph D' is obtained from D by replacing an arc (i, j) by arcs (i, k) and (k, j), for a vertex k not in D. Then D is a CI-digraph if and only if D' is.*

Proof. Suppose a matrix M is a matrix of relations for S_D. Let M' be obtained from M by replacing the column for (i, j) by two copies of the same column, one for (i, k) and one for (k, j). Then M' is a matrix of relations for $S_{D'}$. Similarly, if one starts with a matrix M' of relations for $S_{D'}$, then the columns for (i, k) and (k, j) must be identical and one gets a matrix of relations for S_D by merging the columns into one. Clearly, M is mixed dominating with content 1 if and only if M' is. □

For an affine semigroup S in $\mathbb{Q}^k$ define the *cone* of S to be the set of all nonnegative rational combinations of elements of S. Because S has no invertible elements, this cone is pointed.

A corollary to Theorem 2.3 (Corollary 3.4 of **[6]**) is the following:

Corollary 3.3. *Let S be a d-dimensional affine semigroup that is a complete intersection and suppose that $d \geq 2$. Then the cone of S contains no more than $2d - 2$ extreme rays.*

Proposition 5.1 of **[2]** showed that the extreme rays of the cone of S_D are generated by the vectors $e_i - e_j$ for arcs (i, j) if D is a Hasse diagram. The same proof can be used to show that for general D the extreme rays of S_D are generated by the vectors $e_i - e_j$ for independent arcs (i, j) of D. The dimension of the semigroup S_D is $|V(D)| - c$, where c is the number of connected components of D. (By *connected component* of D we mean a connected component of the undirected graph underlying D.) The following corollary is an immediate consequence.

Corollary 3.4. *Let D be a CI-digraph. The number of independent arcs of D is bounded above by $2|V(D)| - 4$.*

The example of Figure 1, possibly with more directed paths of length 2 from the bottom vertex to the top, shows that this bound is tight. This example actually appeared as Example 3.5 of **[6]**. Similarly defined classes of undirected graphs, studied in **[3]** and **[12]**, have smaller upper bounds on the numbers of arcs.

It should be pointed out that there is always a set of circuits that span W and for which the matrix with those circuits as rows has content 1. Such a set is given by the fundamental set of circuits associated with a spanning forest F of D, i.e. the circuits contained in $F \cup e$ where e runs through the arcs of D not in F. In fact, the matrix with rows corresponding to these circuits would be totally unimodular **[11]**. The matrix may, however, contain a square mixed submatrix.

Example 2. Let the vertices of D be labeled $1, 2, \ldots, m$ and for all $1 \leq i < j \leq m$ let (i, j) be an arc of D. Then the arcs $(i, i+1)$ for $i = 1, 2, \ldots, m-1$ are the independent arcs of D. They form a spanning tree F. There are $\binom{m}{2} - (m-1)$ dependent arcs. The matrix whose rows form the fundamental set of circuits associated with F has the form $[N, I]$, where N is an $(\binom{m}{2} - (m-1)) \times (m-1)$ matrix with nonpositive entries. It is easy to see that the matrix $[N, I]$ is mixed dominating. Because it is totally unimodular, it has content 1, so S_D is a complete intersection. The case $m \geq 5$ shows that D need not be planar if its semigroup ring is a complete intersection. If one replaces each dependent arc by a directed path of length 2, then one gets such an example where the digraph is the Hasse diagram of a poset. The tree F is an example of a depth-first search tree. For any undirected graph G, with no parallel edges, one can label the vertices in the order in which they are found in a depth-first search tree and then direct every arc from the lower labeled vertex to the higher labeled vertex. It follows (see **[13]**) that the depth-first search tree contains all of the independent arcs, and that the the matrix whose rows form the fundamental set of circuits associated with this tree has the form $[N, I]$ with N nonpositive. This recovers the result of Gitler et. al. **[9]** that every graph has an orientation for which the associated semigroup ring is a complete intersection.

Example 3. The digraph with the arcs $(1,3), (3,5), (1,4), (4,5), (2,3), (2,4)$ gives a semigroup that is not a complete intersection. The circuits of the digraph are given by the rows of the following matrix (and their negatives.)

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \end{pmatrix}$$
$$\begin{matrix} (1,3) & (3,5) & (1,4) & (4,5) & (2,3) & (2,4) \end{matrix}$$

The dimension of the cycle space is 2. Every choice of two rows of the matrix gives a 2×6 matrix that contains a 2×2 mixed submatrix. In commutative algebra terms, none of the three binomials in the set $\{x_{13}x_{35} - x_{14}x_{45}, x_{23}x_{35} - x_{24}x_{45}, x_{13}x_{24} - x_{14}x_{23}\}$ is in the ideal generated by the other two. For the partial order $\mathcal{P}$ with Hasse diagram in Figure 2, we get

$$\Psi(\mathcal{P}) = \frac{x_1x_2 - x_3x_4 - x_1x_5 - x_2x_5 + x_3x_5 + x_4x_5}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_5)(x_4-x_5)}.$$

Note that the numerator does not factor.

If we add to this digraph the arc $(1,2)$, then the resulting digraph would be a CI-digraph. Thus the complete intersection property is not always preserved by deletion of arcs.

Reversing the orientation of any of the arcs of Example 3 makes one of the paths from vertex 3 to vertex 4 a directed path, and we will see that this critical change makes the resulting digraph a CI-digraph.

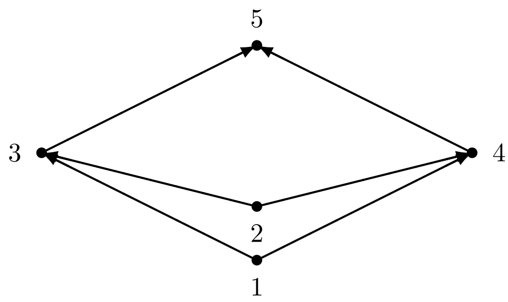

FIGURE 2. Poset for Example 3

4. Deletion and Contraction. In this section, we prove some fundamental tools for analyzing the CI-digraph property. The results involving deletion of dependent arcs will be essential for proving the main results of the paper. Results on contraction of independent arcs, which appear fundamental but are not directly used in later sections, appear at the end of the section.

Lemma 4.1. *Suppose that acyclic digraph D' is obtained from D by adding an arc e which is then dependent in D'. Then D' is a CI-digraph if D is.*

Proof. Suppose D is a CI-digraph. Then there is a basis for the cycle space of D consisting of the rows of a matrix M that is mixed dominating with content one. We can append a column of zeroes corresponding to the dependent arc e to M and then append a row which is $+1$ on the entry in the new column, -1 on each of the entries corresponding to the arcs in the directed path in D from the tail of e to the head of e, and zeroes elsewhere, to get a matrix M'. The matrix M' is clearly mixed dominating. It has content 1 because M does. □

Lemma 4.2. *Suppose that D is a CI-digraph and e is a dependent arc of D from vertex i to vertex j. Suppose that the cycle space of D has a basis consisting of the rows of a matrix M, partitioned as in (1). Then there is, distinct from e, a red directed path or a green directed path from i to j.*

Proof. Without loss of generality, suppose that e is red. Let P be a directed path distinct from e from the tail of e to the head of e. If P contains any green arcs, let β be the sum of the green elements of T corresponding to these arcs. Then β is in the span of the red elements of T, so β is in the semigroup generated by the red elements of T. If β is equal to the element of T corresponding to e, then there is a green path from i to j other than e. Otherwise, there is a red path from i to j other than e. □

Proposition 4.3. *Suppose D is a CI-digraph containing a dependent arc e. Then D has a mixed dominating matrix of relations with content 1 that has a single nonzero entry in the column corresponding to arc e. The entry of nonzero sign in column e is the only entry of its sign in its row.*

Proof. Suppose D is a CI-digraph. Then there is an $r \times n$ matrix M of relations that decomposes as in (1). The proof is by induction on r. The statement is clearly true if $r = 1$. Without loss of generality, let e be a red arc.

Suppose there is a red directed path other than e from the tail of e to the head of e. By the inductive hypothesis, we can replace A by a matrix A' that has a unique nonzero entry in column e and the row containing this entry, call it x, has 1 in column e and no other positive entries. If the bottom row of M has $a_e = 0$, then the matrix M' decomposed as in (1), with the matrix A' replacing A is what we want. If $a_e = 1$, then we can eliminate component e of the last row between row x and the last row to obtain a vector y that is zero on component e. The matrix M'' obtained from M' by replacing the bottom row by y is still mixed dominating with content 1. M'' satisfies the requirements of the Proposition.

Now suppose that e is independent in the red subgraph. The previous lemma shows that there is a green directed path from the tail of e to the head of e. The proof of that lemma shows that the vector a of the decomposition (1) corresponds to a directed path from the tail of e to the head of e. The assumption that e is independent in the red graph means that the vector a in the decomposition (1) has a single nonzero entry corresponding to e. Replace each row of M other than the bottom row that contains a nonzero in entry e by the result of eliminating e between the given row and the bottom row. It is easy to see that M' is mixed dominating with content 1 if M is. M' satisfies the requirements of the Proposition. □

Corollary 4.4. *Suppose that D is an acyclic digraph and e is a dependent arc. Then the digraph D' obtained from D by deleting e is a CI-digraph if and only if D is.*

By deleting dependent arcs from an acyclic digraph one by one until there are no dependent arcs left, one obtains the Hasse diagram of the partial order given by the transitive closure of the digraph. Thus we have the following.

Corollary 4.5. *An acyclic digraph D is a CI-digraph if and only if the Hasse diagram of the partial order given by the transitive closure of D is a CI-digraph.*

Definition 4.6. *Let v be a vertex and C a circuit of a digraph D. Let X be a set of arcs leaving v in D. Then v is an X-source in C if two* *arcs of X are in C. Similarly, if X is a set of arcs entering v in D, then v is an X-sink in C if two arcs of X are in C.*

Proposition 4.7. *Suppose that D is a CI-digraph and that M is a mixed dominating matrix of content 1 whose rows form a cycle basis for D. Suppose v is a vertex of D and X is a set of arcs entering v. Then v is an X-sink in at most $|X| - 1$ circuits given by rows of M. Similarly, if X is a set of arcs leaving v, then v is an X-source in at most $|X| - 1$ circuits given by rows of M.*

Proof. If v is an X-sink in $|X|$ circuits corresponding to rows of M, then M contains a square mixed submatrix in the columns corresponding to X. A similar conclusion follows if v is an X-source in $|X|$ circuits corresponding to rows of M. □

We will call a path P in an acyclic digraph D *dependent* if each of its interior vertices has degree 2 in the graph underlying D, and there is a directed path other than P from one endpoint of P to another. The following Corollary is a special case of Proposition 4.7 corresponding to the case $|X| = 2$. If a dependent path is not a directed path, then at least one of its interior vertices must be an $|X|$-sink or an $|X|$-source for a set X of cardinality 2.

Corollary 4.8. *Suppose that D is a CI-digraph and that P is a dependent path in C that is not a directed path. Let M be a mixed dominating matrix of content 1 whose rows form a cycle basis for D. Then the arcs in P appear in only one of the circuits corresponding to rows of M.*

Proposition 4.9. *Let D be a CI-digraph and let P be a dependent path in D. Let Q be a directed path other than P from one end of P to the other. Then there is a mixed dominating matrix M' with content 1 whose rows, one of which is formed from the circuit containing P and Q, form a cycle basis for D.*

Proof. Let M be a mixed dominating matrix with content 1 whose rows form a cycle basis for D. Let M' be the matrix obtained from M by replacing the row corresponding to a circuit that uses the arcs of

P by the row formed from the circuit involving P and Q. Then M' is mixed dominating with content 1. □

Proposition 4.9 shows that there is some freedom in choosing a directed path Q to go along with a dependent path P in a row of M. Note however that for every choice of Q, the set of sinks and sources in the resulting circuit is the same.

Proposition 4.10. *Suppose D is an acyclic digraph, and suppose D' is obtained from D by removing the arcs and interior vertices of a dependent path P, Then D' is a CI-digraph if and only if D is.*

Proof. If P is a directed path, then we can assume by Lemma 3.2 that P consists of a single arc. Then the Proposition follows from Lemma 4.1 and Proposition 4.3. If P is not a directed path, suppose that D is a CI-digraph let M be a mixed dominating matrix of relations for D with content 1. There may only be one row of M, which we assume to be the bottom row, that contains nonzero entries in the components corresponding to arcs of P. If two rows had such entries, then M would contain a 2×2 mixed submatrix. The matrix obtained from M by deleting the bottom row will be mixed dominating and have content 1. This matrix is therefore a matrix of relations for the digraph D'. Thus D' is a CI-digraph. Now suppose that D' is a CI-digraph and that M' is a mixed dominating basis for the cycle space of D' with content one. Append columns of zeros to M' corresponding to arcs of P, and then append a row that is a circuit consisting of P and a directed path of D' from one endpoint of P to the other. Call the resulting matrix M. Then M is mixed dominating, because all the nonzero entries of the bottom row of M corresponding to the arcs of D' have the same sign. M has content one because M' does. □

If D is an acyclic digraph and e is a dependent arc of D, then contracting e leaves a graph that is no longer acyclic. The situation is different, however, if e is independent.

Proposition 4.11. *Suppose that D is a CI-digraph and that e is an independent arc of D. Then the digraph D' obtained from D by contracting e is a CI-digraph.*

Proof. Suppose that D is a CI-digraph, e is an independent arc of D, and M is a mixed dominating matrix with content 1 whose rows form a cycle basis for D. Let M' be the matrix obtained from M by deleting the column corresponding to arc e. Because e is independent, every row of M' is mixed. Because M was dominating, so is M'. Every row of M' is in the cycle space of the digraph D' obtained from D by contracting e. Thus M' is a cycle basis of D'. M' also has content 1. This is because the rows of M' are linearly independent, and the $r \times r$ nonsingular submatrices of M' are nonsingular submatrices of M and thus all have determinant 1. □

A *chord* of a circuit C in a digraph is an arc that is not in C but has its endpoints in C.

Corollary 4.12. *Suppose that D is a CI-digraph and that C is a circuit of D that has an independent chord. Then the circuit C does not correspond to any of the rows of a minimal matrix of relations for D.*

Proof. Suppose C is a circuit with an independent chord e. If we delete entry e from the vector corresponding to C, we get a vector that is in the cycle space of D' but is not a circuit. If C corresponded to a row of M, then the matrix M' constructed as in the proof of the previous proposition would not have all rows corresponding to circuits. This contradicts Theorem 3.1. □

5. Existence of a Separating Path. A directed path $P = (v_0, a_1, v_1, a_2, \ldots, a_k, v_k)$ in $D = (V, \mathcal{A})$ is called *separating* if there are subdigraphs $D_1 = (V_1, \mathcal{A}_1)$ and $D_2 = (V_2, \mathcal{A}_2)$ of D so that $V = V_1 \cup V_2$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, $V_1 \cap V_2 = \{v_0, v_1, \ldots, v_k\}$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \{a_1, a_2, \ldots, a_k\}$ and such that each of D_1 and D_2 contains an arc not in P. In this case, we say that D is obtained from D_1 and D_2 by gluing along P.

We assume that D is a digraph for which the semigroup S_D is a complete intersection. By the discussion in Section 3, we can also assume that the graph underlying D is 2-connected. This implies that for any two arcs, there exists a circuit containing both. 2-connectivity implies that any separating dipath must contain at least one arc. By Theorem 2.3, we assume that the matrix of relations has the form:

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \\ \hline a & b \end{array}\right]$$

We remind the reader that the arcs corresponding to the columns of A are called *red* and those corresponding to the columns of B *green*. Let $\mathcal{A}_R$ be the set of red arcs, V_R the set of vertices incident to red arcs, and c_R the number of connected components of the digraph $(V_R, \mathcal{A}_R)$. Similarly, let $\mathcal{A}_G$ be the set of green arcs, V_G the set of vertices incident to green arcs, and c_G the number of connected components of the digraph $(V_G, \mathcal{A}_G)$. By assumption, D has one connected component.

The number of rows of A is given by $|\mathcal{A}_R| - |V_R| + c_R$, and the number of rows of B is given by $|\mathcal{A}_G| - |V_G| + c_G$. The number of rows of the whole matrix is given by $|\mathcal{A}| - |V| + 1$. Clearly, we have

$$(|\mathcal{A}_R| - |V_R| + c_R) + (|\mathcal{A}_G| - |V_G| + c_G) + 1 = |\mathcal{A}| - |V| + 1.$$

The sizes of V, V_R and V_G are related by $|V| = |V_R| + |V_G| - |V_R \cap V_G|$. We also have $|\mathcal{A}| = |\mathcal{A}_R| + |\mathcal{A}_G|$. We must therefore have

$$|V_R \cap V_G| - (c_R + c_G) + 1 = 1. \tag{2}$$

A vertex of $(V, \mathcal{A})$ is contained in at most one connected component of $(V_R, \mathcal{A}_R)$ (a *red* component) and at most one connected component of $(V_G, \mathcal{A}_G)$ (a *green* component). Define $\mathcal{C}$ to be the union of the set of connected components of $(V_R, \mathcal{A}_R)$ and the set of connected components of $(V_G, \mathcal{A}_G)$. For each vertex x of $V_R \cap V_G$, define $C_R(x)$ to be the red component containing x and $C_G(x)$ to be the green component containing x. Define the bipartite multigraph Γ to have vertex set $\mathcal{C}$ and edge set $\{\{C_R(x), C_G(x)\} : x \in V_R \cap V_G\}$.

Lemma 5.1. *The graph Γ is connected and contains a unique circuit.*

Proof. Let C_1 and C_2 be vertices of Γ. Let v_1 and v_2 be vertices of D in C_1 and C_2, respectively. Because the graph underlying D is connected, there is a path P from v_1 to v_2 in D. This path corresponds to a path in Γ containing the vertices that contain the vertices of P. Equation (2) can be interpreted as $|E(\Gamma)| - |V(\Gamma)| + 1 = 1$, showing that the cycle space of Γ has dimension 1. Thus Γ contains a unique circuit. $\square$

Let C_{ab} be the circuit of $(V, \mathcal{A})$ given by the row (a, b) in the decomposition (1). There must be an integer $k \geq 1$ and red-green vertices $v_0, v_1, v_2, \ldots, v_{2k} = v_0$ so that for one of the ways of going around C_{ab}, there is a red path in C_{ab} from v_{2i+1} to v_{2i+2} with all arcs directed forward and a green path in C_{ab} from v_{2i} to v_{2i+1} with all arcs backward for $i = 1, 2, \ldots, k-1$. A *maximal monochromatic subpath* of C_{ab} is a a path from v_i to v_{i+1} for some $i \in \{0, 1, \ldots, 2k-1\}$. A maximal monochromatic subpath of C_{ab} will be called red (resp. green) if it contains red (resp. green) edges. If none of the maximal monochromatic subpaths of C_{ab} are in components of $(V_R, \mathcal{A}_R)$ or of $V_G, \mathcal{A}_G)$ that have arcs not in C_{ab}, then C_{ab} is D, because D is 2-connected.

Proposition 5.2. *Suppose that C is a circuit of D. Then either C is contained in a red component or a green component, or C has the same structure as C_{ab}, in that it consists of paths between v_i and v_{i+1} for each $i \in \{0, 1, \ldots, 2k-1\}$.*

Proof. The circuit C naturally defines a cycle in the graph Γ among the vertices of Γ that contain edges of C. Recall that Γ contains a unique cycle, and that the arcs of Γ correspond to red-green vertices of D. Every vertex that is incident to both red and green arcs of C must be v_i for some $i \in \{0, 1, \ldots, 2k-1\}$, by the uniqueness of the circuit in Γ. The uniqueness of this circuit also implies that each v_i appears in C. $\square$

The vertices in C_{ab} that have indegree 0 or 2 in C_{ab} fill a special role, in that every circuit containing red and green arcs must pass through each of these vertices. Another role that they fill is as the nonzero components of the element α of the semigroup S, which is the generator of the one-dimensional subsemigroup that is the intersection of the red and green subsemigroups of S. Let

$$V^+(C_{ab}) = \{i \in V(C_{ab}) : i \text{ has indegree 0 in } C_{ab}\}$$
$$V^-(C_{ab}) = \{i \in V(C_{ab}) : i \text{ has indegree 2 in } C_{ab}\}.$$

Proposition 5.3. *The vector α generating the intersection of the red and green subsemigroups of S_D is equal to $\sum_{i\in V^+(C_{ab})} e_i - \sum_{j\in V^-(C_{ab})} e_j$.*

Lemma 5.4. *Each of the red and green components of D is a CI-digraph.*

Proof. By Proposition 2.3, the digraphs $(V_R, \mathcal{A}_R)$ with red edges and $(V_G, \mathcal{A}_G)$ of green edges are CI-digraphs. Therefore, the connected components of these digraphs are CI-digraphs. □

Corollary 5.5. *The graph Γ consists of one circuit.*

Proof. This follows from proposition 5.2 and the assumption that D is 2-connected. □

Proposition 5.6. *Suppose P is a maximal monochromatic subpath of C_{ab} and P is in a red (resp. green) component of D which contains an arc not in C_{ab}. Then P is a separating directed path.*

Proof. Let $\mathcal{A}_1$ be the set of arcs in the red (resp. green) component of D containing P, and let $\mathcal{A}_2$ be the remaining arcs of D. Let V_1 consist of vertices of D incident to arcs of $\mathcal{A}_1$ and let V_2 consist of vertices of D incident to arcs of $\mathcal{A}_2$. Then the only elements of $V_1 \cap V_2$ are the endpoints of P. □

Referring back to Example 1, the bipartite multigraph Γ contains one vertex for one red component, and one vertex for one green component. It contains two edges, one corresponding to the element 1 which is both in the red component and the green component, and one edge corresponding to the element 6 which is both in the red component and the green component. The cycle C_{ab} is made up of a maximal red chain $(1, 2, 6)$ and a maximal green chain $(6, 5, 1)$. Both of these chains are in components that contain other vertices not in C_{ab}. The chain $(1, 2, 6)$ is in the red component which contains vertex $v = 3$. C_{ab} contains a vertex $w = 5$ that is not in the chain $(1, 2, 6)$. Every path in the Hasse diagram from v to w goes through a vertex of $(1, 2, 6)$.

Theorem 5.7. *Suppose that D is a 2-connected CI-digraph and that P is a directed path in D. Then there is a dependent path in D that does not use any of the arcs in P.*

Proof. The proof is by induction on the dimension of the cycle space. It is clearly true when the dimension is 1. Suppose the dimension of the cycle space is $r \geq 2$ and P is a directed path in D. Suppose that M is a mixed dominating matrix of relations for D with content 1, decomposed as in matrix (1). Consider the circuit C_{ab}. The red and green components of D need not be 2-connected. However, any cut vertices that they have will be on the circuit C_{ab}. Let Y be the set of all red-green vertices, together with all of the cut vertices of the red and green components, ordered cyclically according to their appearance in C_{ab}. We will refer to a subdigraph between two consecutive vertices of Y as a component. Then each component of D between two consecutive vertices of Y is 2-connected or a single arc. Because $r \geq 2$, then at least one of these components will be 2-connected.

Case 1: There is a 2-connected component $\mathcal{C}$ between two consecutive vertices of Y that has no arcs of P. Apply the inductive hypothesis to this component and the path that is the intersection of C_{ab} with $\mathcal{C}$. The resulting path Q that is dependent in $\mathcal{C}$ misses P and is also dependent in D because none of its interior vertices is in Y.

Case 2: P contains arcs of every 2-connected component between consecutive vertices of Y, but P has all arcs of the same color. Let $\mathcal{C}$ be the red component or green component containing P. Each red component and green component other than $\mathcal{C}$ must be a directed path. Together they form a dependent path with no arcs of P.

Case 3: P has arcs in every 2-connected component between consecutive vertices of Y, and one of the interior vertices, call it v, of P is a red-green vertex. Let $\mathcal{C}$ be the red or green component containing v in which v is the head of the arc of C_{ab} in $\mathcal{C}$ containing v and the tail of the arc of P in $\mathcal{C}$ containing v or vice versa. The part of $\mathcal{C}$ between v and the next vertex of Y in $\mathcal{C}$ is a 2-connected component that we call $\mathcal{C}'$. Apply the inductive hypothesis to $\mathcal{C}'$ and the path in $\mathcal{C}'$ that is the intersection of $\mathcal{C}'$ and $C_{ab} \cup P$. The resulting path Q that is dependent in this component is also dependent in D, because none of its interior vertices is in Y. □

Corollary 5.8. *Suppose that D is a 2-connected CI-digraph. Then D has at least 2 dependent paths, neither of which is a subpath of the other.*

Proof. Let P be a dependent path guaranteed by the Theorem, and let P' be a minimal dependent subpath of P. Let P'' be a directed path with at least one arc that is a subpath of P'. Then the dependent path Q that does not contain any arcs of P'' is such that neither of Q nor P' is a subpath of the other. □

Corollary 5.9. *A digraph D is a CI-digraph if and only if there is a sequence $D = D_1, D_2, \ldots, D_r$ where D_r has a single circuit and for $i = 1, 2, \ldots, r-1$, the digraph D_{i+1} is obtained from D_i by removing the arcs and interior vertices of a dependent path.*

Proof. Suppose D is a CI-digraph. Theorem 5.7 shows that $D = D_1$ contains a dependent path in each 2-connected component and Proposition 4.10 shows that the digraph obtained by removing the arcs and interior vertices of the path is a CI-digraph. If D_i is a CI-digraph for some i, and D_i is not a circuit, then, for the same reasons, there will be a dependent path whose removal leaves a CI-digraph D_{i+1}. □

Example 4. In Figure 3 below, the "ears" form the two dependent paths. This example contains no directed dependent paths. By increasing the size of the diamond, one can make the ratio of arcs to vertices arbitrarily close to 2.

Theorem 5.7 and its corollaries suggest an efficient algorithm to recognize CI-digraphs, by locating and removing dependent paths. Each dependent path removed contributes one row to the matrix of relations. In Example 3, notice that there is no dependent path.

Corollary 5.10. *Let D be a CI-digraph and let P be a separating directed path. Then each of the subdigraphs D_1 and D_2 glued together along P to form D is a CI-digraph.*

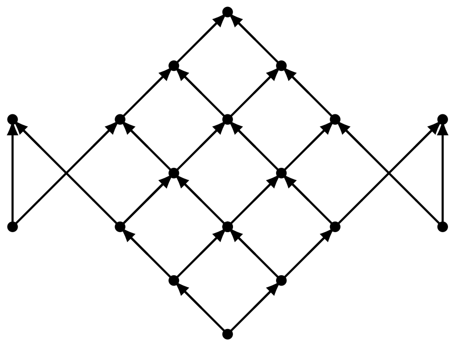

FIGURE 3. Diamond with Ears

Proof. Apply Theorem 5.7 repeatedly to D and the path P, locating and removing dependent paths. Note that none of the dependent paths to be removed may have a vertex of P as an interior vertex, so each of the dependent paths is contained in D_1 or D_2. If a directed path Q from one end of a dependent path R to the other contains arcs in both D_1 and D_2 that are not in P, then one can also obtain, by replacing subpaths of this path by subpaths of P, a directed path Q' from one end of R to the other that has all its arcs in the same component, D_1 or D_2, that contains R. By Proposition 4.9, we can use this circuit in a mixed dominating cycle basis. Each path yields a circuit in D_1 or in D_2. The circuits in D_1 form a mixed dominating matrix with content 1 that is a cycle basis for D_1 and the circuits in D_2 form a mixed dominating matrix with content 1 that is a cycle basis for D_2. □

Definition 5.11. *Let the rows of an $r \times n$ matrix M span the cycle space of D. The rows of M form a weakly fundamental cycle basis if the rows and columns of M can be permuted so that M contains an $r \times r$ nonsingular triangular submatrix.*

Theorem 5.12. *Suppose that M is a mixed dominating matrix of relations of content 1 for the CI-digraph D. Then the rows of M form a weakly fundamental cycle basis of the cycle space of D.*

Proof. We proceed by induction on r, the number of rows of M. The result is clear if $r = 1$. Suppose the Theorem is true whenever the matrix has $r - 1$ rows. Let M have r rows. Let e be an arc that only appears in one row of M. Permute the rows of M so that the row containing a nonzero entry in column e is last. Permute the columns of M so that the columns containing entries in this row that do not have nonzero entries in any other row are last. The remaining rows of M form a mixed dominating matrix of content 1, which is a matrix of relations for the semigroup of the digraph with arc e deleted. By induction, the rows and columns of this smaller matrix can be permuted to make the matrix contain a nonsingular $(r-1)\times(r-1)$ lower triangular submatrix. □

6. A non-TU example. The following example shows that the matrix M for a complete intersection need not be totally unimodular. It is reminiscent of Figure 23 of [**1**].

Example 5.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}$$
$$\begin{array}{cccccccccccc} (1,2) & (2,3) & (3,4) & (4,5) & (1,8) & (8,4) & (2,9) & (9,5) & (2,6) & (6,4) & (1,7) & (7,5) \end{array}$$

The submatrix in rows 1,2 and 4 and columns $(1,2),(2,3)$ and $(4,5)$ has determinant 2. Note also that columns $(1,8),(2,9),(2,6)$ and $(1,7)$ contain a triangular basis with -1s on the diagonal.

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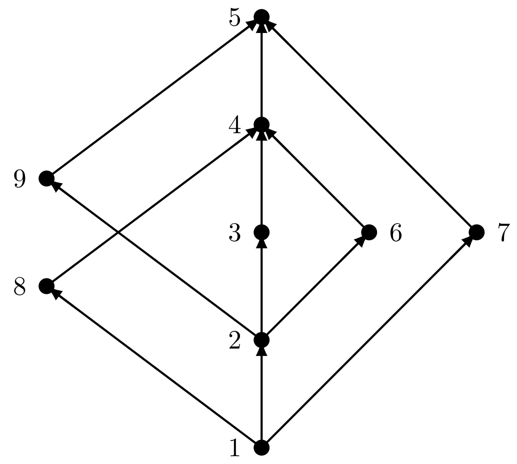

FIGURE 4. Poset for Example 5

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