Complete Intersection Affine Semigroup Rings Arising from Posets

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Dedicated to the memory of Professor Klaus G. Fischer

Abstract

We apply theorems of Fischer, Morris and Shapiro on affine semigroup rings to show that if a certain affine semigroup ring defined by a poset is a complete intersection, then the poset is either unicyclic or contains a chain, the removal of which increases the number of connected components of the Hasse diagram. This is the converse of a theorem of Boussicault, Feray, Lascoux and Reiner [2]. We show that the rows of a matrix of relations for the affine semigroup form a weakly fundamental cycle basis consisting of circuits of the digraph given by the Hasse diagram of the poset, but give an example of such a matrix that is not totally unimodular. We also show that the number of edges of the Hasse diagram of a poset for which the affine semigroup ring is a complete intersection is bounded above by twice the number of vertices minus 4.

1 Introduction

The present work applies graph theory to answer a question about partially ordered sets, in the spirit of the recent paper by Boussicault and Feray [1]. Ring theory is used in the introduction to define a particular class of partially ordered sets, but will not be encountered in later sections.

Suppose that $T$ is a set of $n$ nonzero vectors in $\mathbb{Q}^m$, and let $S$ be the semigroup generated by $T$. If $S$ contains no invertible elements and if the group generated by $T$ has rank $d$, then $S$ is called an affine semigroup of dimension $d$. One may associate to $S$ the space of relations $W$ over the rationals $\mathbb{Q}$. If $\{u_1, u_2, \ldots, u_r\}$ is a set of integral vectors that forms a basis for $W$ over the rationals, we will call the $r \times n$ matrix with rows $u^T_1, u^T_2, \ldots, u^T_r$ a matrix of relations for $W$. A vector $u$ in $\mathbb{Z}^n$ is called mixed if it has

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both positive and negative components. An $r \times n$ matrix is called \textit{mixed} if each of its rows is mixed. It is called \textit{dominating} if it does not contain a square mixed submatrix. The \textit{content} of an $r \times n$ matrix $M$ is the gcd of all of the $r \times r$ minors of $M$.

A useful fact about mixed dominating matrices is the following (Corollary 2.8 of [4]):

\textbf{Proposition 1} Let $M$ be an $r \times n$ mixed dominating matrix. Then the rows of $M$ are linearly independent.

Let $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ be the polynomial ring in the variables $X_1, X_2, \ldots, X_n$ over the integers $\mathbb{Z}$. For any vector $u$ in $\mathbb{Z}^n$, define the vectors $u^+$ and $u^-$ by $u_i^+ = \max\{0, u_i\}$ and $u_i^- = \max\{0, -u_i\}$ for $i = 1, 2, \ldots, n$. Define $X^u = \prod_{i=1}^n X_i^{u_i}$. The semigroup ring $\mathbb{Z}[S]$ is $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ modulo the ideal of relations $\{X^u^+ - X^u^-\}$ where $u$ is any integral vector in $W$. This ideal is prime and has height $\dim_{\mathbb{Q}} W = r$ (see [3]). Hence, if this ideal is generated by $r$ elements, then the semigroup $\mathbb{Z}[S]$ is called a complete intersection. In this case, we will also say that $S$ is a complete intersection. The following theorem (Corollary 2.10 of [4]) characterizes complete intersection affine semigroup rings in terms of the existence of a matrix of relations with certain properties.

\textbf{Theorem 1} Let $S$ be a finitely generated subsemigroup of $\mathbb{Z}^q$ that contains no invertible elements. Then $S$ is a complete intersection if and only if there exists an integral basis of the relation space of $S$ whose coefficient matrix is dominating with content 1.

The paper [4] proved that an $r \times n$ matrix $M$ with integer entries and rows $u_1, u_2, \ldots, u_r$ is dominating if and only if for every integer linear combination $w$ of the rows of $M$ there is a $j$ in $\{1, 2, \ldots, r\}$ so that $w^+ \geq u_j^+$. This illustrates the origin of the term \textit{dominating}. In this case, one can multiply $X^{u_j^+}$ by a monomial in $\mathbb{Z}[X_1, X_2, \ldots, X_n]$ to get $X^{w^+}$. The matrix $M$ has content 1 if and only if $W \cap \mathbb{Z}^n$ equals the set of integer linear combinations of the rows of $M$.

The mixed dominating property of a matrix was shown to be equivalent to a certain block structure of the matrix in Theorem 2.2 of [5] (see also [6]):

\textbf{Theorem 2} Let $M$ be a mixed dominating $r \times (r + d)$ matrix with $r > 0$. Then there is a rearrangement of the rows and columns of $M$ so that the resulting matrix has the form

$$
\begin{bmatrix}
A & 0 \\
0 & B \\
a & b
\end{bmatrix}
$$

(1)

where $A$ and $B$ are mixed dominating matrices of sizes $t \times (t + d_1)$ and $s \times (s + d_2)$ respectively, with $t \geq 0, s \geq 0$ and $s + t + 1 = r$ and $d_1 + d_2 - 1 = d$. Additionally, $a$ and $b$ are $1 \times (t + d_1)$ and $(1 \times (s + d_2)$ nonzero, nonmixed matrices respectively of opposite sign.

We apply these theorems to affine semigroup rings $\mathbb{Z}[S_P]$ arising from posets $P$. Our main goal is the following result, one direction of which was already proved in [2]:

\textbf{Theorem 3} An affine semigroup ring $\mathbb{Z}[S_P]$ is a complete intersection if and only if the poset $P$ can be obtained from unicyclic posets by repeated gluings along chains.
Section 2 introduces the affine semigroup ring \( \mathbb{Z}[S_P] \). It is shown in this section that if the semigroup ring is a complete intersection, then the rows of the matrix of relations must correspond to circuits of the digraph given by the Hasse diagram of the poset. Here it is also shown that the number of edges of the Hasse diagram of \( P \) if \( \mathbb{Z}[S_P] \) is a complete intersection is bounded above by \( 2|P| - 4 \). Section 3 establishes the existence of a separating chain in the poset if the affine semigroup is a complete intersection and the poset is not unicyclic. Section 4 shows that the semigroup rings given by the two posets that are glued together on the separating chain must themselves be complete intersections. This will conclude the proof of Theorem 3. Section 5 applies the decomposition to show that the rows of the matrix of relations must be a weakly fundamental cycle basis for the digraph given by the Hasse diagram. An example which shows that the matrix need not be totally unimodular is also given in this section. Finally, Section 6 discusses the strongly planar case and gives an example of a poset for which the semigroup is a complete intersection but the undirected graph underlying the Hasse diagram is not planar.

2 Affine semigroup rings from posets

Suppose \( P \) is a finite partially ordered set (poset) with elements labelled by \( \{1, 2, \ldots, m\} \). The function

\[
\Psi(P) = \sum_{w \in \mathcal{L}(P)} \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{m-1}} - x_{w_m})}
\]

where \( \mathcal{L}(P) \) is the set of linear extensions of \( P \), was studied by Boussicault and Feray [1]. There it was shown that this function could be written as a rational function

\[
\Psi(P) = \frac{N(P)}{\prod_{i<j}(x_i - x_j)},
\]

where \( i \prec j \) means that \( i \) is covered by \( j \). That paper investigated the question: When does the numerator \( N(P) \) factor?. They showed that if the poset \( P \) can be obtained from smaller posets \( P_1 \) and \( P_2 \) by gluing along a chain (See section 3) then \( N(P) \) is the product of \( N(P_1) \) and \( N(P_2) \). The investigations of [1] were inspired by a theorem of Greene, [9], which showed that \( N(P) \) is a product of linear factors if the poset \( P \) is strongly planar (See section 6). The paper left open the problem of determining if a decomposition of \( N(P) \) into linear factors implies that \( P \) can be decomposed into smaller posets \( P_1 \) and \( P_2 \) which can be glued along a chain to form \( P \).

For \( i = 1, 2, \ldots, m \), let \( e_i \) be the \( i \)th standard basis vector of \( \mathbb{R}^m \) Let \( T_P \) be the set of vectors \( \{e_i - e_j : i \prec j\} \), corresponding to the covering relations of \( P \), and let \( S_P \) be the semigroup generated by \( T_P \). The paper [2] showed that \( N(P) \) factors into a product of linear factors if the semigroup ring \( \mathbb{Z}[S_P] \) is a complete intersection. There it was also shown that if \( P \) is obtained from smaller posets \( P_1 \) and \( P_2 \) by gluing along a chain, and if \( \mathbb{Z}[S_{P_1}] \) and \( \mathbb{Z}[S_{P_2}] \) are complete intersections, then so is \( \mathbb{Z}[S_P] \). This in particular showed that for strongly planar \( P \), the ring \( \mathbb{Z}[S_P] \) is a complete intersection. The main point of our paper is to show that if a poset has a complete intersection semigroup ring, then either each connected component of its Hasse diagram has at most one circuit, or it is obtained from smaller posets by gluing along a chain.
We will think of the Hasse diagram of $P$ as a directed graph $D_P$, with all edges directed from the smaller to the larger element. Because the elements of $T_P$ correspond to edges of the Hasse diagram of $P$, the coordinates of the vectors in $W$ correspond to edges of $D_P$. We will refer to elements of $W$ as cycles and will call $W$ the cycle space of $D_P$. From graph theory (the reference [10] provides all of the graph theoretic background needed for this paper), we know that the vectors in $W$ of minimal support arise from circuits of $D_P$. For each circuit of $D_P$, pick an orientation of the circuit and let $u$ be the vector that is $+1$ if the edge is traversed upward in the orientation of the circuit, $-1$ if the edge is traversed downward in the orientation, and $0$ otherwise. It is also known that every integer vector in $W$ can be written conformally as a nonnegative rational combination of circuit vectors, that is, a linear combination in which every nonzero component of each of the summands agrees in sign with the corresponding component of the sum.

Boussicault et. al. [2] proved that $\mathbb{Z}[S_P]$ is a complete intersection if and only if the semigroup rings corresponding to each of the biconnected components of the Hasse diagram of $P$ is a complete intersection. It was also proved in [2] that if the Hasse diagram of $P$ is a circuit, then $\mathbb{Z}[S_P]$ is a complete intersection. These results can also be established from the decomposition in Theorem 2. If one wants to determine if $\mathbb{Z}[S_P]$ is a complete intersection, then, it is sufficient to consider the poset with all edges removed that do not appear in a circuit of $P$. We will call a poset unicyclic if each of its connected components contains at most one cycle.

Semigroup rings for general directed graphs were studied by Gitler et. al. [7], [8]. Directed graphs defined by Hasse diagrams have the property that for every circuit, there are at least two edges directed compatibly with each of the two possible orientations of the circuit. That the directed graph is acyclic (has no circuit with all edges pointed upward) comes from the transitive property of a poset. This property is enough to show that every element of the cycle space is a mixed vector, and that the semigroup has no invertible elements. If there were a circuit with only one edge pointing up and all others pointing down, or vice versa, then the relation represented by the up edge would be implied by the relations represented by the rest of the circuit. The Hasse diagram does not contain any relations that are implied by other relations.

**Theorem 4** Suppose that $\mathbb{Z}[S_P]$ is a complete intersection affine semigroup ring and let $M$ be a coefficient matrix of a basis of the relation space of $S_P$ that is mixed dominating with content 1. Then the rows of $M$ are circuits.

**Proof.** If the relation space is 1-dimensional, then it is spanned by a circuit, and the entries of a circuit are $\pm 1$. Now assume that the relation space is $r$-dimensional for $r > 1$, and that the matrix of the basis is decomposed as in Theorem 2 and has content 1. We can assume by induction that the rows of $A$ and $B$ are circuits. Suppose that the vector $u_r = (a, b)$ is not a circuit. Let $x$ be a circuit that conforms to $(a, b)$. Then $x$ is in $W$ and must be an integral combination $x = a_1 u_1 + a_2 u_2 + \cdots + a_r u_r$ of the rows of $M$. The vector $a_r u_r - x$ either conforms to $u_r$ or to the negative of $u_r$. The vector $a_r u_r - x$ is not the zero vector, because then $u_r$ would be a circuit. Thus the matrix obtained from $M$ by replacing $u_r$ by $a_r u_r - x$ is still dominating. Because its rows are linearly dependent, we have a contradiction to Proposition 1.

The conclusion of Theorem 4 holds for any semigroup $S$ given by an acyclic directed graph, not necessarily one determined by a Hasse diagram. It was shown in [7] that
the ideal of relations for $\mathbb{Z}[S]$ is generated by binomials corresponding to circuits for any semigroup $S$ given by a directed graph.

For an affine semigroup $S$ in $\mathbb{Q}^k$ define the cone of $T$ to be the set of all positive rational combinations of elements of $S$. Because $S$ has no invertible elements, this cone is pointed. A corollary to Theorem 2 (Corollary 3.4 of [5]) is the following:

**Corollary 1** Let $S$ be a $d$-dimensional affine semigroup that is a complete intersection and suppose that $d \geq 2$. Then the cone of $S$ contains no more than $2d - 2$ extreme rays.

Proposition 5.1 of [2] showed that the extreme rays of the cone of $S_P$ are generated by the vectors $e_i - e_j$ for edges $(i,j)$ of the Hasse diagram $D_P$. The dimension of the semigroup $S_P$ is $|P| - c$, where $c$ is the number of connected components of $D_P$. The following corollary is an immediate consequence.

**Corollary 2** Let $P$ be a finite poset and let $S_P$ be its affine semigroup. If $S_P$ is a complete intersection, then the number of edges of the Hasse diagram is bounded above by $2|P| - 4$.

This bound shows that the directed graphs coming from Hasse diagrams are a very small subset of the set of all directed graphs. Compare to section 4.1 of [8], where it is shown that for every connected graph $G$ there is an orientation of $G$ so that the resulting semigroup ring is a complete intersection.

**Example 1** (See Figure 1.) Suppose the elements of the poset $P$ are $1, 2, 3, 4, 5, 6$ and the covering relations are $(1,2), (2,6), (1,3), (3,6), (1,4), (4,6), (1,5), (5,6)$. Then $1$ is a minimum element and $6$ is a maximum element. A matrix of relations is (with the columns labelled in the bottom row.)

$$
\begin{bmatrix}
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\
(1,2) & (2,6) & (1,3) & (3,6) & (1,4) & (4,6) & (1,5) & (5,6)
\end{bmatrix}
$$

This matrix has content 1 and decomposes as in equation (1), so the semigroup is a complete intersection. Note that one could replace the last row by $(1,1,1,1,-1,-1,-1,-1)$ and get another set of rows that spans the cycle space and for which the matrix decomposes as in equation 1. However, the resulting matrix would have content 2.

It should be pointed out that there is always a set of circuits that span $W$ and for which the matrix with those circuits as rows has content 1. Such a set is given by the fundamental set of circuits associated with a spanning forest $F$ of $D_P$, i.e. the circuits contained in $F \cup e$ where $e$ runs through the edges of $D_P$ not in $F$. In fact, the matrix with rows corresponding to these circuits would be totally unimodular [10]. The matrix may, however, contain a square mixed submatrix.

**Example 2** The poset with the covering relations $(1,3), (3,5), (1,4), (4,5), (2,3), (2,4)$ gives a semigroup that is not a complete intersection. The circuits of the Hasse diagram of this poset are given by the rows of the following matrix (and their negatives.)

$$
\begin{bmatrix}
1 & 1 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & 1 & 0 & -1 & 1 & -1 & -1 \\
(1,3) & (3,5) & (1,4) & (4,5) & (2,3) & (2,4)
\end{bmatrix}
$$
The dimension of the cycle space is 2. Every choice of two rows of the matrix gives a $2 \times 6$ matrix that contains a $2 \times 2$ mixed submatrix. For this $P$, we get $\Psi(P) = \frac{x_1x_2 - x_3x_4 - x_1x_5 - x_2x_5 + x_3x_5 + x_4x_5}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_5)(x_4 - x_5)}$. Note that the numerator does not factor. The affine semigroup ring would still not be a complete intersection if we added elements 6 between 1 and 3 and 7 between 1 and 4, subdividing the edges $(1, 3)$ and $(1, 4)$ of the Hasse diagram. If we then add to this poset the relation $1 \leq 2$, then the resulting poset would have a complete intersection semigroup. (The poset is strongly planar.) Thus the complete intersection property is not preserved by deletion of edges.
3 Existence of a Separating Chain

By a separating chain of \( P \) we mean a directed path \( T \) in the Hasse diagram of \( P \) for which the removal of \( T \) increases the number of connected components of the Hasse diagram. Suppose \( P_1 \) and \( P_2 \) are subposets of \( P \) whose intersection is \( T \). If \( P \) is the union of \( P_1 \) and \( P_2 \), \( P_1 \setminus T \) and \( P_2 \setminus T \) are nonempty, and the set of covering relations of \( P \) is the union of the sets of covering relations of \( P_1 \) and \( P_2 \), \( P \) is said to be obtained from \( P_1 \) and \( P_2 \) by gluing along \( T \).

In this and the following section, we assume that \( P \) is a poset for which the semigroup \( S_P \) is a complete intersection. By Theorem 2, we assume that \( M \) is a matrix of relations of the form:

\[
M = \begin{bmatrix}
A & 0 \\
0 & B \\
a & b
\end{bmatrix}
\]

We call the edges corresponding to the columns of \( A \) red and those corresponding to the columns of \( B \) green. Let \( E_1 \) be the set of red edges, \( V_1 \) the set of vertices incident to red edges, and \( c_1 \) the number of connected components of the digraph \((V_1, E_1)\). Similarly, let \( E_2 \) be the set of green edges, \( V_2 \) the set of vertices incident to green edges, and \( c_2 \) the number of connected components of the digraph \((V_2, E_2)\). Define \( E \) to be the set of all edges of \( D_P \), let \( V \) be the vertex set of \( D_P \), and let \( c \) be the number of connected components of \( D_P = (V, E) \). We will assume that every edge of \( E \) is contained in some cycle of \( D_P \) and that \( D_P \) has no isolated vertices.

The number of rows of \( A \) is given by \(|E_1| - |V_1| + c_1\), and the number of rows of \( B \) is given by \(|E_2| - |V_2| + c_2\). The number of rows of the whole matrix is given by \(|E| - |V| + c\).

Clearly, we have

\[
(|E_1| - |V_1| + c_1) + (|E_2| - |V_2| + c_2) + 1 = |E| - |V| + c.
\]

The sizes of \( V, V_1 \) and \( V_2 \) are related by \(|V| = |V_1| + |V_2| - |V_1 \cap V_2|\). We also have \(|E| = |E_1| + |E_2|\). We must therefore have

\[
|V_1 \cap V_2| - (c_1 + c_2) + c = 1. \tag{2}
\]

A vertex of \((V, E)\) is contained in at most one connected component of \((V_1, E_1)\) (a red component) and at most one connected component of \((V_2, E_2)\) (a green component). Define \( C \) to be the union of the set of connected components of \((V_1, E_1)\) and the set of connected components of \((V_2, E_2)\). For each vertex \( x \) of \( V_1 \cap V_2 \), define \( C_1(x) \) to be the red component containing \( x \) and \( C_2(x) \) to be the green component containing \( x \). Define the bipartite multigraph \( \Gamma \) to have vertex set \( C \) and edge set \( \{C_1(x), C_2(x)\} : x \in V_1 \cap V_2\} \).

**Lemma 1** The number of connected components of \( \Gamma \) equals the number of connected components of \((V, E)\). The graph \( \Gamma \) contains a unique cycle.

**Proof.** By a path of \( D_P \) we mean a path in the undirected graph graph underlying \( D_P \), so the edges in a path of \( D_P \) do not have to be directed in the same direction as the path. Let \( C \) be a connected component of \( D_P \). Define \( \alpha(C) \) to be the union of the
set of red and green components that contain edges of $C$. Then $\alpha(C)$ is a subset of the
vertex set of $\Gamma$. If $C^1$ and $C^2$ are elements of $\alpha(C)$, then we can let $v$ be a vertex of $C^1$
and let $w$ be a vertex of $C^2$. There is a path $\pi$ from $v$ to $w$ in $D_P$. This path induces a
path $(C^1 = C_1, C_2, \ldots, C_\ell = C^2)$ in $\Gamma$ from $C_1$ to $C_2$ as follows. Let $C_2$ be the component
of the first edge of the path $\pi$ that is not in $C_1$. The vertex of $\pi$ preceding this edge is
in $V_1 \cap V_2$ and so there is an edge in $\Gamma$ from $C_1$ to $C_2$. Let $C_3$ be the component of the
first edge of $\pi$ that is not in $C_2$, etc. Thus $\alpha(C)$ is contained in a connected component
of $\Gamma$. Similarly, if $(C_1, C_2, \ldots, C_\ell)$ is a path in $\Gamma$, $v$ is a vertex of $C_1$ and $w$ is a vertex of
$C_\ell$, then there is a path $\pi$ in $D_P$ that goes from $v$ in $C_1$ to a vertex in $C_1 \cap C_2$ and then
within $C_2$ to a vertex in $C_2 \cap C_3$, etc. Thus $\alpha(C)$ is a component of $\Gamma$. We have shown
that $\alpha$ is one to one. It is onto because the $E$ is the union of $E_1$ and $E_2$.

Equation (2) shows that the cycle space of $\Gamma$ has dimension 1, so $\Gamma$ contains a unique
cycle.

Let $C$ be the circuit of $(V, E)$ given by the row $(a, b)$ in the decomposition (1). The red edges in $C$
are directed upward in the poset, and the green edges in $C$ are directed downward. A maximal monochromatic subpath of $C$ is a maximal sequence
$T = (v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1})$ such that $e_1, e_2, \ldots, e_k$ are all of the same color. A maximal
monochromatic subpath may contain just one vertex if that vertex is incident to an edge of
one color that is not in $C$, but the two edges of $C$ that are incident to the vertex are of
the other color. A maximal monochromatic subpath of $C$ will be called red (resp. green)
if it contains red (resp. green) edges. If it contains no edges, then it will be called red
(resp. green) if it is incident to red (resp. green) edges not in the path but the path edges
incident to it are green (resp. red). If none of the maximal monochromatic subpaths of
$C$ are in components of $(V_1, E_1)$ or of $(V_2, E_2)$ that have vertices not in $C$, then $C$ is a
component of $D_P$ by itself. If this is not the case, then we can find a chain that separates
$P$ into nonempty components.

**Proposition 2** Suppose that a red maximal monochromatic subpath $T$ of $C$ is in a connected component of $(V_1, E_1)$ that contains a vertex $v$ that is not in $C$. Let $w$ be a vertex of $C$ that is not in $T$. Then every path in $V$ from $v$ to $w$ goes through a vertex of $T$.

**Proof.** The circuit $C$ naturally defines a cycle in the graph $\Gamma$ among the vertices of $\Gamma$ that contain edges of $C$. Recall that $\Gamma$ contains a unique cycle, and that the edges of $\Gamma$ correspond to red-green vertices of $D_P$. Suppose that $v$ is in a component of $(V_1, E_1)$ that contains a maximal monochromatic subpath $T$ of $C$, but that $v$ is not in $C$. Suppose that there is a path $(v = v_1, v_2, \ldots, v_k = w)$ in $D_P$ from $v$ to a vertex $w$ so that $v_1, v_2, \ldots, v_{k-1}$ are not in $C$ and $w$ is in $C$ but not in $T$. (There must be such a vertex $w$ because $C$ has at least two green edges and at least two red edges.) We claim that this path violates the uniqueness of the cycle in $\Gamma$. Let $x$ be a vertex of $T$ that is in the same component of $(V_1, E_1)$ as $v$. The path from $v$ to $w$ can be extended to a cycle $\tilde{C}$ by going from $w$ back to $x$ one way around the circuit $C$ (and then back to $v$ within $(V_1, E_1)$), and to a cycle $\tilde{C}$ by going from $w$ to $x$ the other way around $C$. At least one of the cycles $\tilde{C}$ and $\tilde{C}$ is missing a red-green vertex of $C$, so the cycle in $\Gamma$ corresponding to such a cycle is not the same as the cycle corresponding to $C$.

We illustrate this proposition with Example 1. The bipartite multigraph $\Gamma$ contains
one vertex for one red component, and one vertex for one green component. It contains
two edges, one corresponding to the element 1 which is both in the red component and the green component, and one edge corresponding to the element 6 which is both in the red component and the green component. The cycle $C$ is made up of a maximal red chain $(1, 2, 6)$ and a maximal green chain $(6, 5, 1)$. Both of these chains are in components that contain other vertices not in $C$. The chain $(1, 2, 6)$ is in the red component which contains vertex $v = 3$. $C$ contains a vertex $w = 5$ that is not in the chain $(1, 2, 6)$. Every path in the Hasse diagram from $v$ to $w$ goes through a vertex of $(1, 2, 6)$.

**Example 3** $P$ has elements 1, 2, 3, 4, 5, 6, 7, 8, and the covering relations are $(2, 6), (3, 7), (4, 8), (1, 5), (2, 5), (1, 6), (1, 7), (3, 5), (1, 8), (4, 5)$.

![Figure 3: Poset for Example 3](image)

Note that here the matrix “$B$” is empty, i.e. it has no rows. The multigraph $\Gamma$ has four vertices. There is one red component containing edges $(2, 6), (3, 7), (1, 5), (2, 5), (1, 6), (1, 7), (3, 5)$ and another red component containing only the edge $(4, 8)$. One green component contains only the edge $(1, 8)$ and the other contains the edge $(4, 5)$. The multigraph $\Gamma$ is just a 4-cycle. The cycle $C$ consists of two red chains and two green chains, with each chain containing only one edge. Only one of these chains, $(1, 5)$, is in a component that contains vertices not in the chain. For example, $v = 2$ is in the red component containing the chain $(1, 5)$. Every path in the Hasse diagram from $v = 2$ to $w = 4$ must intersect the chain $(1, 5)$.

The undirected graph underlying Example 3 is an example of a ring graph, studied by Gitler et. al. [7], [8]. It can be constructed by gluing circuits along a single edge. Gitler et. al. showed that the semigroup rings for any orientation of such graphs were complete intersections.
4 Decomposing a matrix of relations

In this section, we will prove that each of the semigroups formed by a connected component of \((V, E) \setminus T\) together with \(T\) is a complete intersection. This is a converse of Theorem 8.6 of [2]. The proof of that theorem showed that one can use the relations from the component semigroups to get a set of relations for \((V, E)\).

Let us assume that the relation space for \(S_P\) is spanned by the rows of a mixed dominating matrix \(M\), decomposed as (1). We will assume that the Hasse diagram is not a cycle. For simplicity, we will assume that it is connected. As in the previous section, we assume that the circuit \(C\) corresponds to the last row of \(M\), and that \(T\) is a red maximal monochromatic subpath of \(C\) that is in a red connected component that contains a vertex that is not in \(C\).

**Lemma 2** No two maximal monochromatic subpaths of \(C\) belong to the same connected component of \((V_1, E_1)\).

**Proof.** Suppose there were vertices \(w_1\) and \(w_2\) of \(C\) that were in different maximal monochromatic subpaths of \(C\) but in the same connected component of \((V_1, E_1)\). In that case, there would be a red-green vertex \(z\) of \(C\) on one of the paths in \(C\) from \(w_1\) to \(w_2\). The cycle from \(w_1\) to \(w_2\) through the red component and then back to \(w_1\) on the path of \(C\) not containing \(z\) gives a cycle in \(\Gamma\) that misses the edge corresponding to \(z\). This cycle is different from that corresponding to \(C\).

We could also conclude in the same way that no two maximal monochromatic subpaths of \(C\) belong to the same connected component of \((V_2, E_2)\). A maximal red monochromatic subpath may, however, contain some single-vertex green maximal monochromatic subpaths.

The deletion of \(T\) from the graph \(D_P\) leaves a disconnected graph. Let \(F\) be the connected component of \(D_P \setminus T\) that contains \(C \setminus T\), and let \(\overline{F}\) be \((D_P \setminus T) \setminus F\). Note that any cycle of \(D_P\) that contains vertices of \(F\) and \(\overline{F}\) must contain vertices of \(T\). Because \(T\) is in a red component, there will be no red cycles that contain vertices of \(F\) and vertices of \(\overline{F}\). We will now see that there are green cycles that contain vertices of both \(F\) and \(\overline{F}\) only in a special case.

**Lemma 3** If \(C\) has more than one maximal green connected component that contains edges of \(C\), then there is no green cycle that contains vertices of \(F\) and vertices of \(\overline{F}\).

**Proof.** Any green cycle that contains vertices of both \(F\) and \(\overline{F}\) must contain two vertices of \(T\). Any single-vertex green maximal monochromatic subpath contained in \(T\) would be different from either of the green maximal monochromatic subpaths that are attached to the endpoints of \(T\). In order for the green cycle to meet two vertices of \(T\), the green maximal monochromatic subpaths that are attached to the endpoints of \(T\) must be the same path.

**Lemma 4** Suppose that there is a green connected component \(G\) which contains vertices in both \(F\) and \(\overline{F}\). Let \(R\) be the red connected component that contains \(T\). Then \(G \cap \overline{F}\) and \(R \cap \overline{F}\) are in different connected components of \(\overline{F}\).
Proof. If there were a path from a vertex of $G \cap F$ to a vertex of $R \cap \overline{F}$ within $F$, then this path would contain a red-green vertex $z$. This path could be extended to a cycle containing $T$, and this cycle would correspond to a cycle of $\gamma$ that contains the edge corresponding to $z$ that would be different from the cycle of $\gamma$ corresponding to $C$. 

**Proposition 3** The rows and columns of $M$ can be rearranged so that it has the form

$$
\begin{bmatrix}
A' & B' & 0 \\
0 & C' & D'
\end{bmatrix},
$$

where the columns of $B'$ correspond to the edges of $T$.

Proof. If there is no green circuit that contains vertices in both $F$ and $\overline{F}$, then it is clear that the rows and columns of $M$ can be permuted to get a matrix of the form of the proposition, where the columns of $A'$ correspond to edges of $F$ and edges connecting $F$ to $T$, and the columns of $D'$ correspond to edges of $\overline{F}$ and edges connecting $\overline{F}$ to $T$. If there is a green circuit containing vertices of both $F$ and $\overline{F}$, then move the intersection of the green connected component containing this circuit with $\overline{F}$ to $F$. The matrix can then be rearranged as in the Proposition.

**Example 1, continued.** (See Figure 1.) The cycle $C$ consists of the red chain $(1, 2, 6)$, which is $T$, and the green chain $(6, 5, 1)$. Thus we originally put vertices 3 and 4 in $F$ and vertex 5 in $\overline{F}$. Because vertex 4 is in a green component that contains vertices of both $F$ and $\overline{F}$, we move vertex 4 from $F$ to $\overline{F}$. We rearrange the columns of $M$ so that those corresponding to edges incident to 3 come first. We do not need to rearrange the rows of $M$.

$$
\begin{bmatrix}
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\
(1, 3) & (3, 6) & (1, 2) & (2, 6) & (1, 4) & (4, 6) & (1, 5) & (5, 6)
\end{bmatrix}
$$

**Proposition 4** Suppose that $\mathbb{Z}[S_P]$ is a complete intersection and that the matrix of relations $M$ is decomposed as in (3). Then the matrices $[A' | B']$ and $[C' | D']$ are mixed dominating with content 1.

Proof. Because the rearrangement of rows and columns of the matrix $M$ cannot create any new square mixed submatrices, the matrices $[A' | B']$ and $[C' | D']$ are mixed dominating. To show that $[A' | B']$ has content 1, we have to show that every circuit using only edges corresponding to columns of $A'$ and $B'$ is an integer linear combination of the rows of $A'$ and $B'$ and that every circuit using only edges corresponding to columns of $C'$ and $D'$ is an integer linear combination of the rows of $C'$ and $D'$. Let $\hat{C}$ be such a circuit. If it contains only red edges, then it is in a single connected component of $(V_1, E_1)$. From the proof of the last proposition, we see that a connected component of $(V_1, E_1)$ must have all of its edges corresponding to the columns of $[A' | B']$ or all its edges corresponding to columns of $[C' | D']$. Because the semigroup corresponding to the subgraph of red edges is a complete intersection, $\hat{C}$ must be an integer linear combination of rows of $M$ corresponding to the subgraph of red edges, i.e. in the matrix.
A of the decomposition (1). These rows must be in the same connected component of \((V_1, E_1)\), so they must only include edges that correspond to columns of \([ A' \] \(B' \]) or only edges that correspond to columns of \([ C' \] \(D' \]). The proof is similar if the cycle \(\hat{C}\) only contains green edges. Now suppose that \(\hat{C}\) contains both green and red edges. Then we know that it goes through the same sequence of red-green vertices that \(C\) does. Between two red-green vertices it stays inside a connected component (\(V_1, E_1\)) or \((V_2, E_2)\). We know from the proof of the previous proposition that these connected components have all of their edges corresponding to columns of \([ C' \] \(D' \]), with the exception of the red component containing \(T\). Suppose that \(\hat{T}\) is a monochromatic subpath of \(\hat{C}\) between two red-green vertices of \(\hat{C}\), and that \(\hat{T}\) is the monochromatic subpath of \(C\) between those two vertices. Then going forward on \(\hat{T}\) and backward on \(\hat{T}\) gives us a cycle \(\hat{C}\) in the connected component of, say, \((V_1, E_1)\), that contains \(\hat{T}\). In the case that \(\hat{T} = T\), we also get \(\hat{T} = T\), so the cycle \(\hat{C}\) is the zero vector. This cycle \(\hat{C}\) is an integer combination of rows of \(M\) that correspond to the same component of \((V_1, E_1)\). Thus the part of \(\hat{C}\) in this monochromatic subpath is an integer combination of the part of \(C\) in this monochromatic subpath and the rows of \(M\) that make up \(\hat{C}\). We can repeat this for all of the monochromatic subpaths of \(\hat{C}\) to express \(\hat{C}\) as an integer combination of rows of \([ C' \] \(D' \)].

We are now in a position to prove Theorem 3. Recall that a poset is unicyclic if each of the connected components of its Hasse diagram has at most one cycle.

**Proof.** The paper [2] showed that if \(Z[S_P]\) for a unicyclic \(P\) is a complete intersection, and that if \(P\) can be obtained from smaller posets \(P_1\) and \(P_2\) for which \(Z[S_{P_1}]\) and \(Z[S_{P_2}]\) are complete intersections by gluing along a chain, then \(Z[S_P]\) is a complete intersection. Our Proposition 2 shows that if \(Z[S_P]\) is a complete intersection and \(P\) is not unicyclic, then a separating chain exists. Then Proposition 4 shows that the posets that are glued together on this separating chain to form \(P\) are themselves complete intersections.

## 5 A Consequence of the Decomposition

**Proposition 5** Suppose that \(Z[S_P]\) is a complete intersection, with matrix of relations \(M\). There are edges \(e\) and \(f\) of \(P\), not both part of any chain, such that each of \(e\) and \(f\) is in exactly one of the circuits corresponding to rows of \(M\).

**Proof.** The proof is by induction on \(r\). If \(r = 1\), \(P\) consists of one circuit. Fix an orientation of the circuit and let \(e\) be one of the up edges and let \(f\) be one of the down edges. Now suppose that the Proposition is true whenever the matrix of relations has \(r - 1\) rows, and let \(M\) have \(r > 1\) rows. Then there are posets \(P_1\) and \(P_2\), both complete intersections with relation spaces of dimension smaller than \(r\), that can be glued together on a chain \(T\). Let \(e_1\) and \(f_1\) satisfy the Proposition with respect to \(P_1\) and let \(e_2\) and \(f_2\) satisfy it with respect to \(P_2\). If neither \(e_1\) nor \(f_1\) is in \(T\), we can let \(e = e_1\) and \(f = f_1\). If neither \(e_2\) nor \(f_2\) is in \(T\), we can let \(e = e_2\) and \(f = f_2\). If \(e_1\) and \(e_2\) are both in \(T\), then we can let \(e = f_1\) and \(f = f_2\).

**Definition 1** The rows of \(M\) form a weakly fundamental cycle basis if the rows and columns of \(M\) can be permuted so that \(M\) contains an \(r \times r\) nonsingular triangular submatrix.
Theorem 5 Suppose that $M$ is a mixed dominating matrix of relations of content 1 for the complete intersection affine semigroup ring $\mathbb{Z}[S_P]$. Then the rows of $M$ form a weakly fundamental cycle basis.

Proof. We proceed by induction on $r$, the number of rows of $M$. The result is clear if $r = 1$. Suppose the Theorem is true whenever the matrix has $r - 1$ rows. Let $M$ be a mixed dominating matrix of relations of content 1, with $r$ rows, for the complete intersection affine semigroup ring $\mathbb{Z}[S_P]$. Let $e$ be an edge that only appears in one row of $M$. Permute the rows of $M$ so that the row containing a nonzero entry in column $e$ is last. Permute the columns of $M$ so that the columns containing entries in this row that do not have nonzero entries in any other row are last. The remaining rows of $M$ form a mixed dominating matrix of content 1, which is a matrix of relations for the semigroup of the poset with edge $e$ deleted. By induction, the rows and columns of this smaller matrix can be permuted to make the matrix contain a nonsingular $(r - 1) \times (r - 1)$ lower triangular submatrix. Take the columns of the larger matrix corresponding to the columns of this submatrix and add column $e$. This will form a nonsingular $r \times r$ lower triangular submatrix. 

The following example shows that the matrix $M$ for a complete intersection need not be totally unimodular. It is reminiscent of Figure 23 of [1].

Example 4

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
(1,2) & (2,3) & (3,4) & (4,5) & (1,8) & (8,4) & (2,9) & (9,5) & (2,6) & (6,4) & (1,7) & (7,5)
\end{bmatrix}
\]
The submatrix in rows 1, 2 and 4 and columns (1, 2), (2, 3) and (4, 5) has determinant 2. Note also that columns (1, 8), (2, 9), (2, 6) and (1, 7) contain a triangular basis with -1s on the diagonal.

## 6 Planarity

A poset $P$ is strongly planar if the poset $P \cup \{\hat{0}, \hat{1}\}$ obtained by adjoining a minimum and maximum element to $P$ has a planar embedding in which the edges of the Hasse diagram are directed upward. Of examples 1–4, only example 1 is strongly planar. The theorem of Greene, reinterpreted in [1] and [2], states that if $P$ is strongly planar, then the set of circuits corresponding to the regions of the above embedding bounded by edges of $P$ gives a set of binomials that generate the affine semigroup ring $\mathbb{Z}[S_P]$. It would be interesting to see a direct proof of the dominating property of such a matrix of relations, one that does not involve the decomposition theorem.

The following procedure shows that this set of circuits forms a weakly fundamental cycle basis. Take an edge $f$ that is on the boundary of the planar embedding of the Hasse diagram and add it to $B$. Remove from $D_p$ the edges of the Hasse diagram that are on the boundary of $D_p$ and on the bounded face that contained $f$. Continue taking edges from the boundary of the Hasse diagram and adding them to $B$, until there are no more bounded regions. The $r \times r$ submatrix of $M$ indexed by these edges can have its rows and columns reordered to make a triangular matrix with $\pm 1$ on the diagonal.

The matrix $M$ containing the face circuits is totally unimodular, as is shown in Lemma 14 of [10]. We sketch the argument here. Assume that the circuits are obtained from traversing them clockwise. The matrix $M$ can then be seen as the incidence matrix for the directed planar dual graph, with arcs going from left to right. This matrix has at most two nonzero entries per column. If it has two entries in a column, one is 1 and the other is -1. This is sufficient to show total unimodularity.

**Example 5** Recall from Corollary 2 that all planar graphs $(V, E)$ that contain no triangles must also satisfy the inequality $|E| \leq 2|V| - 4$. The graphs underlying Examples 1–4 are all planar. It would be incorrect, however, to infer that the graphs underlying the Hasse diagrams (which are also triangle free) of posets for which the semigroups $S_P$ are complete intersections are all planar. Consider the poset with Hasse diagram in Figure 5.

The poset is strongly planar, so its semigroup is a complete intersection. Now insert an element 11 which covers element 4 and is covered by element 2 and is in no other covering relations. In the enlarged poset, one can color the edges of the Figure red and color the new edges (4, 11) and (11, 2) green. If one enlarges the matrix of relations for the poset of the Figure by a row corresponding to the circuit $(((4, 1), (1, 2), (2, 11), (11, 4)))$, one has a decomposition as in Theorem 2. It is also possible to find a 6 $\times$ 6 submatrix of the matrix of relations that has determinant with magnitude 1. One such submatrix has columns indexed by (1, 6), (4, 9), (2, 8), (2, 3), (3, 7), (1, 4). Thus the semigroup for the larger poset is also a complete intersection. The graph underlying the Hasse diagram is not planar, however, because it is a subdivision of $K_5$. 
7 Directions for further research

We do not yet know if factorization into linear terms of the numerator $N(P)$ of the function $Ψ_P$ from the introduction implies that $\mathbb{Z}[S_P]$ is a complete intersection. We would like to have a better algorithm to show that an affine semigroup ring $\mathbb{Z}[S_P]$ is not a complete intersection. The algorithm used in Example 2, listing the circuits and showing that no set of $r$ circuits yielded a matrix that is mixed dominating and has content 1, seems very inefficient.

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References


