The Jensen-Pólya Program for the Riemann Hypothesis and Related Problems

Ken Ono (U of Virginia)
Riemann’s zeta-function

**Definition (Riemann)**

For $\text{Re}(s) > 1$, define the zeta-function by

$$
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.
$$
Hyperbolicity of Jensen polynomials

Introduction

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Theorem (Fundamental Theorem)

1. The function $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ (apart from a simple pole at $s = 1$ with residue 1).
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**Theorem (Fundamental Theorem)**

1. The function \( \zeta(s) \) has an analytic continuation to \( \mathbb{C} \) (apart from a simple pole at \( s = 1 \) with residue 1).

2. We have the functional equation

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \cdot \zeta(1 - s).
\]
Hilbert’s 8th Problem

**Conjecture (Riemann Hypothesis)**

Apart from negative evens, the zeros of $\zeta(s)$ satisfy $\text{Re}(s) = \frac{1}{2}$. 


Introduction

Hilbert’s 8th Problem

Conjecture (Riemann Hypothesis)

Apart from negative evens, the zeros of \( \zeta(s) \) satisfy \( \text{Re}(s) = \frac{1}{2} \).

“Without doubt, it would be desirable to have a rigorous proof of this proposition; however, I have left this research...because it appears to be unnecessary for the immediate goal of my study....”

Bernhard Riemann (1859)
IMPORTANT REMARKS

Fact (Riemann’s Motivation)

Proposed RH because of Gauss’ Conjecture that $\pi(X) \sim \frac{X}{\log X}$. 

What is known?

1. The first “gazillion” zeros satisfy RH (van de Lune, Odlyzko).
2. $\geq 41\%$ of zeros satisfy RH (Selberg, Levinson, Conrey,...).
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JENSEN-PÓLYA PROGRAM

J. W. L. Jensen
(1859–1925)

George Pólya
(1887–1985)
JENSEN-PÓLYA PROGRAM

DEFINITION

The **Riemann Xi-function** is the entire function

\[
\Xi(z) := \frac{1}{2} \left( -z^2 - \frac{1}{4} \right) \pi^{i z - \frac{1}{4}} \Gamma \left( -\frac{iz}{2} + \frac{1}{4} \right) \zeta \left( -iz + \frac{1}{2} \right).
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The **Riemann Xi-function** is the entire function

\[ \Xi(z) := \frac{1}{2} \left( -z^2 - \frac{1}{4} \right) \pi^{\frac{i}{2} - \frac{1}{4}} \Gamma \left( -\frac{iz}{2} + \frac{1}{4} \right) \zeta \left( -iz + \frac{1}{2} \right). \]

**Remark**

RH is true \( \iff \) all of the zeros of \( \Xi(z) \) are purely real.
Roots of Deg 100 Taylor Poly for $\Xi \left( \frac{1}{2} + z \right)$
Hyperbolicity of Jensen polynomials

Introduction

Roots of Deg 200 Taylor Poly for $\Xi\left(\frac{1}{2} + z\right)$
Roots of Deg 400 Taylor Poly for $\Xi \left( \frac{1}{2} + z \right)$
Takeaway about Taylor Polynomials

- Red points are good approximations of zeros of $\Xi \left( \frac{1}{2} + z \right)$. 

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Takeaway about Taylor Polynomials

- **Red points** are good approximations of zeros of $\Xi \left( \frac{1}{2} + z \right)$.
- The “spurious” **blue** points are annoying.
- As $d \to +\infty$ the spurious points become more prevalent.
**JENSEN POLYNOMIALS**

**Definition (Jensen)**

The **degree** $d$ and **shift** $n$ **Jensen polynomial** for an arithmetic function $a : \mathbb{N} \mapsto \mathbb{R}$ is

$$J_{a,d,n}^d(X) := \sum_{j=0}^{d} a(n + j) \binom{d}{j} X^j$$

$$= a(n + d)X^d + a(n + d - 1)dX^{d-1} + \cdots + a(n).$$
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**Definition**

A polynomial $f \in \mathbb{R}[X]$ is **hyperbolic** if all of its roots are real.
JENSEN’S CRITERION

**Theorem (Jensen-Pólya (1927))**

If \( \Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1 - s) \),

What was known?
The hyperbolicity for all \( n \) is known for \( d \leq 3 \) by work of Csordas, Norfolk and Varga, and Dimitrov and Lucas.
JENSEN’S CRITERION

**Theorem (Jensen-Pólya (1927))**

If $\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1 - s)$, then define $\gamma(n)$ by

$$(-1 + 4z^2) \Lambda\left(\frac{1}{2} + z\right) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} \cdot z^{2n}.$$
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**RH is equivalent to the hyperbolicity of all of the \( J_{\gamma}^{d,n}(X) \).**
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RH is equivalent to the hyperbolicity of all of the \( J_{d,n}^{\gamma}(X) \).

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New Theorems

“Theorem 1” (Griffin, O, Rolen, Zagier)

For each $d$ at most finitely many $J_d^{n,\gamma}(X)$ are not hyperbolic.
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For each $d$ at most finitely many $J^{d,n}_\gamma(X)$ are not hyperbolic.

Theorem (O+)

Heights $T \ RH \implies$ hyperbolicity of $J^{d,n}(X)$ for all $n$ if $d \ll T^2$. 
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Height $T \RH \implies$ hyperbolicity of $J_{d,n}^{X}$ for all $n$ if $d \ll T^2$.

In particular, $J_{\gamma}^{d,n}(X)$ is hyperbolic for all $n$ when $d \leq 10^{20}$. 

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In particular, $J_{d,n}^\gamma(X)$ is hyperbolic for all $n$ when $d \leq 10^{20}$.

**Theorem (O+)**

If $n \gg 3^d \cdot d\frac{25}{8}$, then $J_{d,n}^\gamma(X)$ is hyperbolic.
Some Remarks

Remarks

1. Offers new evidence for RH.
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1. Offers new evidence for RH.

2. We “locate” the real zeros of the $J_{d,n}^\gamma(X)$. 
Some Remarks

Remarks

1. Offers new evidence for RH.
2. We “locate” the real zeros of the $J_{d,n}^{\gamma}(X)$.
3. Wagner has extended the 1st theorem to other $L$-functions.
**Hermite Polynomials**

**Definition**

The (modified) **Hermite polynomials**

\[
\{ H_d(X) : d \geq 0 \}
\]

are the orthogonal polynomials with respect to \( \mu(X) := e^{-\frac{x^2}{4}} \).
**Hermite Polynomials**

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**Example (The first few Hermite Polynomials)**

\[ H_0(X) = 1 \]
\[ H_1(X) = X \]
\[ H_2(X) = X^2 - 2 \]
\[ H_3(X) = X^3 - 6X \]
**Lemma**

The Hermite polynomials satisfy:

1. Each $H_d(X)$ is hyperbolic with $d$ distinct roots.
2. If $S_d$ denotes the "suitably normalized" zeros of $H_d(X)$, then $S_d \rightarrow$ Wigner's Semicircle Law.
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   $$S_d \rightarrow \text{Wigner’s Semicircle Law}.$$
Theorem 1 (Griffin, O, Rolen, Zagier)

The renormalized Jensen polynomials \( \hat{J}^{d,n}_\gamma(X) \) satisfy

\[
\lim_{n \to +\infty} \hat{J}^{d,n}_\gamma(X) = H_d(X).
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The renormalized Jensen polynomials $\hat{J}^{d,n}_\gamma(X)$ satisfy
$$\lim_{n \to +\infty} \hat{J}^{d,n}_\gamma(X) = H_d(X).$$

For each $d$ at most finitely many $J^{d,n}_\gamma(X)$ are not hyperbolic.
# Degree 3 Normalized Jensen Polynomials

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{J}_{\gamma}^{3,n}(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$\approx 0.9769X^3 + 0.7570X^2 - 5.8690X - 1.2661$</td>
</tr>
<tr>
<td>200</td>
<td>$\approx 0.9872X^3 + 0.5625X^2 - 5.9153X - 0.9159$</td>
</tr>
<tr>
<td>300</td>
<td>$\approx 0.9911X^3 + 0.4705X^2 - 5.9374X - 0.7580$</td>
</tr>
<tr>
<td>400</td>
<td>$\approx 0.9931X^3 + 0.4136X^2 - 5.9501X - 0.6623$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$10^8$</td>
<td>$\approx 0.9999X^3 + 0.0009X^2 - 5.9999X - 0.0014$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$H_3(X) = X^3 - 6X$</td>
</tr>
</tbody>
</table>
Random Matrix Model Predictions

Freeman Dyson
Hugh Montgomery
Andrew Odlyzko

Ken Ono (U of Virginia)
Random Matrix Model Predictions

Gaussian Unitary Ensemble (GUE) (1970s)

The nontrivial zeros of $\zeta(s)$ appear to be “distributed like” the eigenvalues of random Hermitian matrices.
Relation to our work

“Theorem” (Griffin, O, Rolen, Zagier)

GUE holds for Riemann’s $\zeta(s)$ in derivative aspect.
**Relation to our work**

"Theorem" (Griffin, O, Rolen, Zagier)

*GUE holds for Riemann’s $\zeta(s)$ in derivative aspect.*

**Sketch of Proof**

1. The $J_{\gamma,n}^d(X)$ model the zeros of the $n$th derivative $\Xi^{(n)}(X)$. 

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Relation to our work

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1. The $J_{d,n}^\gamma(X)$ model the zeros of the $n$th derivative $\Xi^{(n)}(X)$.
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1. The $J_{d,n}^{\gamma}(X)$ model the zeros of the $n$th derivative $\Xi^{(n)}(X)$.
2. The derivatives are predicted to satisfy GUE.
3. For fixed $d$, we proved that

$$\lim_{n \to +\infty} \tilde{J}_{\gamma}^{d,n}(X) = H_d(X).$$
Relation to our work

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Sketch of Proof

1. The $J_{\gamma}^{d,n}(X)$ model the zeros of the $n$th derivative $\Xi^{(n)}(X)$.
2. The derivatives are predicted to satisfy GUE.
3. For fixed $d$, we proved that
   \[
   \lim_{n \to +\infty} \hat{J}_{\gamma}^{d,n}(X) = H_d(X).
   \]
4. The zeros of the $\{H_d(X)\}$ and the eigenvalues in GUE both satisfy Wigner’s Semicircle Distribution.
**Theorem (Pustylnikov (2001), Coffey (2009))**

As \( n \to +\infty \), we have

\[
\xi^{(2n)}(1/2) = \frac{(2n)(2n - 1)(2n - 2)^{-1/4}}{2^{2n-2} \ln(2n-2)^{1/4}} \left[ \ln \left( \frac{2n - 2}{\pi} \right) - \ln \ln \left( \frac{2n - 2}{\pi} \right) + o(1) \right]^{2n-3/2} \\
\times \exp \left( -\frac{2n - 2}{\ln(2n-2)} \right).
\]
Computing derivatives is not easy

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Remarks

1. Derivatives essentially drop to 0 for “small” $n$ before exhibiting exponential growth.
Our Results on RH

Computing derivatives is not easy

**Theorem (Pustylnikov (2001), Coffey (2009))**

As $n \to +\infty$, we have

\[
\xi^{(2n)}(1/2) = \frac{(2n)(2n-1)(2n-2)^{\frac{-1}{4}}}{2^{2n-2} \ln(2n-2)^{\frac{1}{4}}} \left[ \ln \left( \frac{2n-2}{\pi} \right) - \ln \ln \left( \frac{2n-2}{\pi} \right) + o(1) \right]^{2n^{3/2}} \times \exp \left( -\frac{2n-2}{\ln(2n-2)} \right).
\]

**Remarks**

1. Derivatives essentially drop to 0 for "small" $n$ before exhibiting exponential growth.
2. This is insufficient for approximating $J_{\gamma,n}^d(X)$.  

Ken Ono (U of Virginia)  |  Hyperbolicity of Jensen polynomials
**First 10 Taylor coefficients of $\Xi(x)$**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\hat{b}_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.214 009 727 353 926 $(-2)$</td>
</tr>
<tr>
<td>1</td>
<td>7.178 732 598 482 949 $(-4)$</td>
</tr>
<tr>
<td>2</td>
<td>2.314 725 338 818 463 $(-5)$</td>
</tr>
<tr>
<td>3</td>
<td>1.170 499 895 698 397 $(-6)$</td>
</tr>
<tr>
<td>4</td>
<td>7.859 696 022 958 770 $(-8)$</td>
</tr>
<tr>
<td>5</td>
<td>6.474 442 660 924 152 $(-9)$</td>
</tr>
<tr>
<td>6</td>
<td>6.248 509 280 628 118 $(-10)$</td>
</tr>
<tr>
<td>7</td>
<td>6.857 113 566 031 334 $(-11)$</td>
</tr>
<tr>
<td>8</td>
<td>8.379 562 856 498 463 $(-12)$</td>
</tr>
<tr>
<td>9</td>
<td>1.122 895 900 525 652 $(-12)$</td>
</tr>
<tr>
<td>10</td>
<td>1.630 766 572 462 173 $(-13)$</td>
</tr>
</tbody>
</table>
NOTATION

1. We let \( \theta_0(t) := \sum_{k=1}^{\infty} e^{-\pi k^2 t} \),
**ARBITRARY PRECISION ASYMPTOTICS FOR \( \Xi^{(2n)}(0) \)**

**NOTATION**

1. We let \( \theta_0(t) := \sum_{k=1}^{\infty} e^{-\pi k^2 t} \), and define

\[
F(n) := \int_{1}^{\infty} (\log t)^n t^{-3/4} \theta_0(t) \, dt.
\]
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2. Following Riemann, we have

\[
\Xi^{(n)}(0) = (-1)^{n/2} \cdot \frac{32 \binom{n}{2} F(n - 2) - F(n)}{2^{n+2}}
\]
**Notation**

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   \]

3. Let \( L = L(n) \approx \log \left( \frac{n}{\log n} \right) \) be the unique positive solution of the equation \( n = L \cdot (\pi e^L + \frac{3}{4}) \).
**Theorem (Griffin, O, Rolen, Zagier)**

To all orders, as $n \to +\infty$, there are $b_k \in \mathbb{Q}(L)$ such that

$$F(n) \sim \sqrt{2\pi} \frac{L^{n+1}}{\sqrt{(1 + L)n - \frac{3}{4}L^2}} e^{L/4-n/L+3/4} \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots \right),$$

where $b_1 = \frac{2L^4 + 9L^3 + 16L^2 + 6L + 2}{24 (L+1)^3}$. 

**Remarks**

1. Using two terms (i.e. $b_1$) suffices for our RH application.
2. Analysis + Computer $\Rightarrow$ hyperbolicity for $d \leq 10$.

Ken Ono (U of Virginia)
\textbf{THEOREM (GRiffin, O, ROLEn, ZAGIER)}

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\textit{where } b_1 = \frac{2L^4 + 9L^3 + 16L^2 + 6L + 2}{24(L+1)^3}.

\textbf{REMARKS}

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**Theorem (Griffin, O, Rolen, Zagier)**

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**Remarks**

1. Using two terms (i.e. $b_1$) suffices for our RH application.
2. Analysis + Computer $\implies$ hyperbolicity for $d \leq 10^{20}$. 
**Example:** $\hat{\gamma}(n) := \text{TWO-TERM APPROXIMATION}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{\gamma}(n)$</th>
<th>$\gamma(n)$</th>
<th>$\gamma(n)/\hat{\gamma}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\approx 1.6313374394 \times 10^{-17}$</td>
<td>$\approx 1.6323380490 \times 10^{-17}$</td>
<td>$\approx 1.000613367$</td>
</tr>
<tr>
<td>100</td>
<td>$\approx 6.5776471904 \times 10^{-205}$</td>
<td>$\approx 6.5777263785 \times 10^{-205}$</td>
<td>$\approx 1.000012038$</td>
</tr>
<tr>
<td>1000</td>
<td>$\approx 3.8760333086 \times 10^{-2567}$</td>
<td>$\approx 3.8760340890 \times 10^{-2567}$</td>
<td>$\approx 1.000000201$</td>
</tr>
<tr>
<td>10000</td>
<td>$\approx 3.5219798669 \times 10^{-32265}$</td>
<td>$\approx 3.5219798773 \times 10^{-32265}$</td>
<td>$\approx 1.000000002$</td>
</tr>
<tr>
<td>100000</td>
<td>$\approx 6.3953905598 \times 10^{-397097}$</td>
<td>$\approx 6.3953905601 \times 10^{-397097}$</td>
<td>$\approx 1.000000000$</td>
</tr>
</tbody>
</table>
How do these asymptotics imply Theorem 1?
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Theorem 1 is an example of a general phenomenon!
Hyperbolicity of Jensen polynomials
Hermite Distributions

**Remark**

Hyperbolicity of “generating polynomials” is studied in enumerative combinatorics in connection with log-concavity

\[ a(n)^2 \geq a(n - 1)a(n + 1). \]
Hyperbolicity of Jensen polynomials

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Hyperbolic Polynomials in Mathematics

**Remark**

Hyperbolicity of “generating polynomials” is studied in enumerative combinatorics in connection with log-concavity

\[ a(n)^2 \geq a(n - 1)a(n + 1). \]

- Group theory (lattice subgroup enumeration)
- Graph theory
- Symmetric functions
- Additive number theory (partitions)
- ...
**Definition**

A real sequence $a(n)$ has **appropriate growth** if

$$a(n + j) \sim a(n) e^{A(n) j - \delta(n) j^2}$$

for each $j$ for real sequences $\{A(n)\}$ and $\{\delta(n)\} \to 0$. 

What do we mean?

For fixed $d$ and $0 \leq j \leq d$, as $n \to +\infty$ we have

$$\log \frac{a(n + j)}{a(n)} = A(n) j - \delta(n) j^2 + \sum_{i=0}^{d} o_{i,d}(\delta(n)) j^i + O_{d}(\delta(n)^{d+1})$$
**Appropriate Growth**

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for each \( j \) for real sequences \( \{A(n)\} \) and \( \{\delta(n)\} \to 0 \).
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**What do we mean?**

*For fixed $d$ and $0 \leq j \leq d$, as $n \to +\infty$ we have*
**APPROPRIATE GROWTH**

**Definition**
A real sequence \(a(n)\) has **appropriate growth** if

\[
a(n + j) \sim a(n) e^{A(n)j - \delta(n)^2 j^2} \quad (n \to +\infty)
\]

for each \(j\) for real sequences \(\{A(n)\}\) and \(\{\delta(n)\} \to 0\).

**What do we mean?**
*For fixed \(d\) and \(0 \leq j \leq d\), as \(n \to +\infty\) we have*

\[
\log \left( \frac{a(n + j)}{a(n)} \right) = A(n)j - \delta(n)^2 j^2 + \sum_{i=0}^{d} o_{i,d}(\delta(n)^i) j^i + O_d \left( \delta(n)^{d+1} \right).
\]
Hyperbolicity of Jensen polynomials
Hermite Distributions

**GENERAL THEOREM**

**DEFINITION**

If \( a(n) \) has appropriate growth, then the renormalized Jensen polynomials are defined by

\[
\hat{J}_{d,n}^{a,n}(X) := \frac{1}{a(n) \cdot \delta(n)^d} \cdot J_{d,n}^{a,n} \left( \frac{\delta(n)X - 1}{\exp(A(n))} \right).
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**GENERAL THEOREM (Griffin, O, Rolen, Zagier)**

*Suppose that $a(n)$ has **appropriate growth**.*
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**GENERAL THEOREM (GRIFFIN, O, ROLEN, ZAGIER)**

Suppose that \( a(n) \) has *appropriate growth*. For each degree \( d \geq 1 \) we have

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\lim_{n \to +\infty} \hat{J}^{d,n}_a(X) = H_d(X).
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**General Theorem**

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For each \( d \) at most finitely many \( J^d_{a,n}(X) \) are not hyperbolic.
Motivation for our work
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Definition

A partition is any nonincreasing sequence of integers.

\[ p(n) := \# \text{partitions of size } n. \]
**Motivation for our work**

**Definition**

A *partition* is any nonincreasing sequence of integers.

\[ p(n) := \#\text{partitions of size } n. \]

**Example**

We have that \( p(4) = 5 \) because the partitions of 4 are

\[ 4, \ 3 + 1, \ 2 + 2, \ 2 + 1 + 1, \ 1 + 1 + 1 + 1. \]
**Log Concavity of \( p(n) \)**

**Example**

The roots of the quadratic \( J_{p,n}^2(X) \) are

\[
-p(n+1) \pm \sqrt{p(n+1)^2 - p(n)p(n+2)}
\]

\[
p(n+2)
\]

It is **hyperbolic** if and only if \( p(n+1)^2 > p(n)p(n+2) \).
LOG CONCAVITY OF $p(n)$

Example

The roots of the quadratic $J_{p}^{2,n}(X)$ are

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It is hyperbolic if and only if $p(n + 1)^2 > p(n)p(n + 2)$.

Theorem (Nicolas (1978), DeSalvo and Pak (2013))

If $n \geq 25$, then $J_{p}^{2,n}(X)$ is hyperbolic.
Chen’s Conjecture

Theorem (Chen, Jia, Wang (2017))

If \( n \geq 94 \), then \( J_p^{3,n}(X) \) is hyperbolic.
**CHEN’S CONJECTURE**

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If \( n \geq 94 \), then \( J_{p}^{3,n}(X) \) is hyperbolic.

**Conjecture (Chen)**

There is an \( N(d) \) where \( J_{p}^{d,n}(X) \) is hyperbolic for all \( n \geq N(d) \).
Hyperbolicity of Jensen polynomials
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Another Application

**CHEN’S CONJECTURE**

**Theorem (Chen, Jia, Wang (2017))**

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There is an \( N(d) \) where \( J_p^{d,n}(X) \) is hyperbolic for all \( n \geq N(d) \).

**Table 1. Conjectured minimal values of \( N(d) \)**

<table>
<thead>
<tr>
<th>( d )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(d) )</td>
<td>1</td>
<td>25</td>
<td>94</td>
<td>206</td>
<td>381</td>
<td>610</td>
<td>908</td>
<td>1269</td>
<td>1701</td>
</tr>
</tbody>
</table>
Our result

Theorem 2 (Griffin, O, Rolen, Zagier)

Chen’s Conjecture is true.
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Remarks

1. The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).
Our result

Theorem 2 (Griffin, O, Rolen, Zagier)

Chen’s Conjecture is true.

Remarks

1. The proof can be refined case-by-case to prove the minimality of the claimed $N(d)$ (Larson, Wagner).

2. This is a consequence of the General Theorem.
MODULAR FORMS

**Definition**

A weight $k$ weakly holomorphic modular form is a function $f$ on $\mathbb{H}$ satisfying:

1. For all \((a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})\) we have 
   $$f(a\tau + b \ c\tau + d) = (c\tau + d)^k f(\tau).$$
2. The poles of $f$ (if any) are at the cusp $\infty$.

Example (Partition Generating Function)

We have the weight $-1/2$ modular form $f(\tau) = \sum_{n=0}^{\infty} p(n) e^{2\pi i \tau (n - 1/24)}$. 

Ken Ono (U of Virginia)
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JENSEN POLYNOMIALS FOR MODULAR FORMS

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Sketch of Proof. Sufficient asymptotics are known for $a_f(n)$ in terms of Kloosterman sums and Bessel functions.
NATURAL QUESTIONS

Question
What is special about the Hermite polynomials?
Natural Questions

**Question**

What is special about the Hermite polynomials?

**Question**

Is there an even more general theorem?
Lemma (Generating Function)

We have that

\[ e^{-t^2 + Xt} =: \sum_{d=0}^{\infty} H_d(X) \cdot \frac{t^d}{d!} = 1 + X \cdot t + (X^2 - 2) \cdot \frac{t^2}{2} + \ldots \]
**Hermite Polynomial Generating Function**

**Lemma (Generating Function)**

We have that

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\]

**Remark**

The rough idea of the proof is to show for large fixed \( n \) that

\[
\sum_{d=0}^{\infty} \hat{J}_{a,n}^d (X) \cdot \frac{t^d}{d!} \approx e^{-t^2 + xt} = e^{-t^2} \cdot e^{Xt}.
\]
More General Theorem
More General Theorem

Definition

A real sequence \(a(n)\) has appropriate growth for a formal power series \(F(t) := \sum_{i=0}^{\infty} c_i t^i\) if

In the Hermite case we have \(E(n) := e^{A(n)}\) and \(F(t) := e^{-t^2}\).

How does the shape of \(F(t)\) impact “limiting polynomials”?
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Some Remarks

Remark (Limit Polynomials)

If $a : \mathbb{N} \mapsto \mathbb{R}$ is appropriate for $F(t)$, then

$$F(-t) \cdot e^{Xt} = \sum_{d=0}^{\infty} \hat{H}_d(X) \cdot \frac{t^d}{d!}.$$
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**Examples (Special Examples)**

Suppose that \( a : \mathbb{N} \rightarrow \mathbb{R} \) has appropriate growth for \( F(t) \).

1. \( F(t) = -t e^{-t} - 1 \Rightarrow \hat{H}_d(X) = B_d(X) \) Bernoulli poly
2. \( F(t) = 2 e^{-t} + 1 \Rightarrow \hat{H}_d(X) = E_d(X) \) Euler poly
3. \( F(t) = e^{-2t} \Rightarrow \hat{H}_d(X) = H_d(X) \) Hermite poly
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Hyperbolicity of Jensen polynomials
Most General Theorem

LOOSE END

**Theorem (O+)**

\[ \text{Height } T \quad \text{RH} \implies \text{hyperbolicity of } J^{d,n}(X) \text{ for all } n \text{ if } d \gg T^2. \]
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In particular, \( J_{\gamma,n}^d(X) \) is hyperbolic for all \( n \) when \( d \leq 10^{20} \).
Theorem (O+)

Height $T$ RH $\implies$ hyperbolicity of $J_{d,n}^n(X)$ for all $n$ if $d \gg T^2$. In particular, $J_{\gamma,n}^d(X)$ is hyperbolic for all $n$ when $d \leq 10^{20}$.

Sketch of Proof.

- Derivatives causes zeros to line up nicely.
Hyperbolicity of Jensen polynomials
Most General Theorem

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In particular, \( J^{d,n}_\gamma(X) \) is hyperbolic for all \( n \) when \( d \leq 10^{20} \).

**Sketch of Proof.**

- Derivatives causes zeros to line up nicely.
- Truth of RH for low height interfaces well with differentiation.
Hyperbolicity of Jensen polynomials
Most General Theorem

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**Theorem (O+)**

If \( n \gg 3^d \cdot d^{25/8} \), then \( J_{\gamma,n}^d(X) \) is hyperbolic.
LOOSE ENDS

**Theorem (O+)**

If $n \gg 3^d \cdot d^{25/8}$, then $J_{d,n}^{d,n}(X)$ is hyperbolic.

**Sketch of Proof.**

Sturm sequence method with our estimates.
The Future
The Future

Definition

A sequence with appropriate growth for $F(t) = e^{-t^2}$ has type $Z : \mathbb{N} \to \mathbb{R}^+$ if $J_{a,n}^d(X)$ is hyperbolic for $n \geq Z(d)$. 

Remarks

1. RH is equivalent to $\gamma(n)$ having type $Z = 0$.
2. For $\gamma(n)$ we have proved that $Z(d) = O(3^d \cdot d^{25/8})$.
3. Have heuristics for $Z(d)$ for modular form coefficients.
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Wrap Up

**Special Case of** $p(n)$
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Speculation (Griffin, O, Rolen, Zagier)

If $n \leq 32$, then we have $Z(d) \sim 10d^2 \log d$. 
Special Case of $p(n)$

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If $n \leq 32$, then we have $Z(d) \sim 10d^2 \log d$.
Does this continue for larger $n$?
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Speculation (Griffin, O, Rolen, Zagier)

If $n \leq 32$, then we have $Z(d) \sim 10d^2 \log d$.
Does this continue for larger $n$?

Evidence

If we let $\hat{Z}(d) := 10d^2 \log d$, then we have

<table>
<thead>
<tr>
<th>$d$</th>
<th>$N(d)$</th>
<th>$\hat{Z}(d)$</th>
<th>$N(d)/\hat{Z}(d)$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1</td>
<td>$\approx 1$</td>
<td>$\approx 1.00$</td>
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<tr>
<td>2</td>
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<td>$\approx 27.72$</td>
<td>$\approx 0.90$</td>
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<td>$\approx 0.93$</td>
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<tr>
<td>32</td>
<td>35627</td>
<td>$\approx 35489.13$</td>
<td>$\approx 1.00$</td>
</tr>
</tbody>
</table>
Our Results

**General Theorem (Griffin, O, Rolen, Zagier)**

If $a(n)$ has appropriate growth, then for $d \geq 1$ we have

$$\lim_{n \to +\infty} \hat{J}_{a}^{d,n}(X) = H_{d}(X).$$

For each $d$ at most finitely many $J_{a}^{d,n}(X)$ are not hyperbolic.
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Most General Theorem (Griffin, O, Rolen, Zagier)

If \( a(n) \) has appropriate growth for \( F(t) = \sum_{i=0}^{\infty} c_i t^i \), then for each degree \( d \geq 1 \) we have

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APPLICATIONS

Hermite Distributions

1. Jensen-Pólya criterion for RH whenever $n \gg 3^d \cdot d^{25/8}$.
2. Jensen-Pólya criterion for RH for all $n$ if $1 \leq d \leq 10^{20}$.
3. Height $T$ RH $\Rightarrow$ Jensen-Pólya criterion for all $n$ if $d \ll T^2$.
4. The derivative aspect GUE model for Riemann’s $\Xi(x)$.
5. Coeffs of suitable modular forms are log concave and satisfy the higher Turán inequalities (e.g. Chen’s Conjecture).
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+ general theory including Bernoulli and Eulerian distributions.