Hearing the Shape of a Locally Symmetric Space and Arithmetic Groups

Andrei S. Rapinchuk
University of Virginia

GMU September 6, 2019

1 Eigenvalue rigidity and hearing the shape of a drum
- Classical vs. Eigenvalue Rigidity

2 Hearing the Shape
- 1-dimensional case
- Flat tori of dimension > 1
- Weyl's Law and its Consequences

3 Locally symmetric spaces
- Laplace-Beltrami operator
- Isospectral non-isometric manifolds
- Our results

1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity

2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences

3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

Classical rigidity

For $i = 1, 2$, let $\mathcal{G}_i$ be a semi-simple Lie group,

let $\Gamma_i \subset \mathcal{G}_i$ be a lattice (or some other "large" subgroup)

Classical rigidity

For $i = 1, 2$, let $\mathcal{G}_i$ be a semi-simple Lie group,

let $\Gamma_i \subset \mathcal{G}_i$ be a lattice (or some other "large" subgroup)

Then (under appropriate assumptions):

a homo/isomorphism $\phi \colon \Gamma_1 \longrightarrow \Gamma_2$ (virtually) *extends* to a homo/isomorphism of Lie groups $\tilde{\phi} \colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$

Classical rigidity

For $i = 1, 2$, let $\mathcal{G}_i$ be a semi-simple Lie group,

let $\Gamma_i \subset \mathcal{G}_i$ be a lattice (or some other "large" subgroup)

Then (under appropriate assumptions):

a homo/isomorphism $\phi \colon \Gamma_1 \longrightarrow \Gamma_2$ (virtually) *extends* to a homo/isomorphism of Lie groups $\tilde{\phi} \colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$

$$\begin{array}{ccc} \mathcal{G}_1 & & \mathcal{G}_2 \\ \cup & & \cup \\ \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \end{array}$$

Classical rigidity

For $i = 1, 2$, let $\mathcal{G}_i$ be a semi-simple Lie group,

let $\Gamma_i \subset \mathcal{G}_i$ be a lattice (or some other "large" subgroup)

Then (under appropriate assumptions):

a homo/isomorphism $\phi\colon \Gamma_1 \longrightarrow \Gamma_2$ (virtually) *extends* to a homo/isomorphism of Lie groups $\tilde{\phi}\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\tilde{\phi}} & \mathcal{G}_2 \\ \bigcup & & \bigcup \\ \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \end{array}$$

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z})$ $(n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z})$ $(n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z})$ $(n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and
- $G \simeq \mathrm{SL}_n$ over $\mathbb{Q}$.

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and
- $G \simeq \mathrm{SL}_n$ over $\mathbb{Q}$.

Thus,

structure of a (higher rank) arithmetic group *determines*

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and
- $G \simeq \mathrm{SL}_n$ over $\mathbb{Q}$.

Thus,

structure of a (higher rank) arithmetic group *determines*

field of definition &

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and
- $G \simeq \mathrm{SL}_n$ over $\mathbb{Q}$.

Thus,

structure of a (higher rank) arithmetic group *determines*

field of definition & *ambient algebraic group* over this field.

- *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

- *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

- **However**,

• *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

• **However**, one should be able to recover such data as

field of definition & ambient algebraic group

• *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

• **However**, one should be able to recover such data as

field of definition & ambient algebraic group

from *any* Zariski-dense subgroup

- *Structural* approach to rigidity does **not** extend to

  arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

- **However**, one should be able to recover such data as

  field of definition & ambient algebraic group

from *any* Zariski-dense subgroup **if** instead of

structural information

- *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

- **However**, one should be able to recover such data as

field of definition & ambient algebraic group

from *any* Zariski-dense subgroup **if** instead of

structural information

one uses information about the *eigenvalues of elements*.

• *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

• **However**, one should be able to recover such data as

field of definition & ambient algebraic group

from *any* Zariski-dense subgroup **if** instead of

structural information

one uses information about the *eigenvalues of elements*.

• We call this phenomenon *eigenvalue rigidity*.

- How do we match the eigenvalues of elements of two Zariski-dense subgroups?

- How do we match the eigenvalues of elements of two Zariski-dense subgroups?

Note that the subgroups may be represented by matrices

of *different sizes*,

- How do we match the eigenvalues of elements of two Zariski-dense subgroups?

Note that the subgroups may be represented by matrices of *different sizes*,

hence their elements may have *different numbers* of eigenvalues.

- How do we match the eigenvalues of elements of two Zariski-dense subgroups?

Note that the subgroups may be represented by matrices of *different sizes*,

hence their elements may have *different numbers* of eigenvalues.

- Why do we care about eigenvalues?

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* (i.e., *diagonalizable*) matrices,

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* (i.e., *diagonalizable*) matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

be their eigenvalues.

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* (i.e., *diagonalizable*) matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable*

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* (i.e., *diagonalizable*) matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable* if $\exists\ a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \mathbb{Z}$

Let F be a field of characteristic zero (in applications, $F = \mathbb{C}$).

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* (i.e., *diagonalizable*) matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable* if $\exists\ a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable*

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

Example. Let

$$A = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/24 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/12 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

(2) Subgroups $\Gamma_1 \subset \mathrm{GL}_{n_1}(F)$ and $\Gamma_2 \subset \mathrm{GL}_{n_1}(F)$ are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

Example. Let

$$A = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/24 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/12 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

Then A and B are *weakly commensurable* because

$$\lambda_1 = 12 = 4 \cdot 3 = \mu_1 \cdot \mu_2 \quad (\text{or} \quad \lambda_1 = \mu_3^{-1}).$$

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate*,

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate*, (i.e., subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable, even up to conjugation).

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate,* (i.e., subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable, even up to conjugation).

The situation changes *dramatically*

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate*, (i.e., subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable, even up to conjugation).

The situation changes *dramatically* **if** one considers

"big" subgroups,

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate*, (i.e., subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable, even up to conjugation).

The situation changes *dramatically* **if** one considers

"big" subgroups,

viz. *Zariski-dense*, and particularly *arithmetic*, subgroups of (almost) simple algebraic groups.

However, no powers A^m and B^n $(m, n \neq 0)$ are *conjugate*, (i.e., subgroups $\langle A \rangle$ and $\langle B \rangle$ are not commensurable, even up to conjugation).

The situation changes *dramatically* **if** one considers

"big" subgroups,

viz. *Zariski-dense*, and particularly *arithmetic*, subgroups of (almost) simple algebraic groups.

- **Reason:** these subgroups contain special elements, called *generic.*

M. Kac, Amer. Math. Monthly,
73(1966), 1-23

M. Kac, Amer. Math. Monthly, **73**(1966), 1-23

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

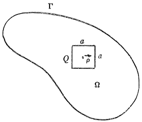


FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane Ω, held fixed along its boundary Γ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a certain constant depending on the physical properties of the membrane and on the tension under which the membrane is held.

I shall choose units to make $c^2 = \frac{1}{2}$.

1

M. Kac, Amer. Math. Monthly, **73**(1966), 1-23

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

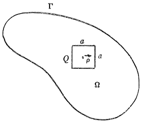


FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane Ω, held fixed along its boundary Γ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a certain constant depending on the physical properties of the membrane and on the tension under which the membrane is held.

I shall choose units to make $c^2 = \frac{1}{2}$.

1

1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity

2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences

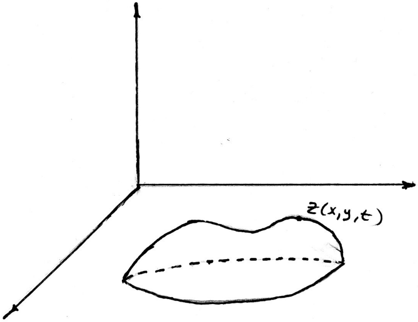
3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

Hearing the Shape

- Consider a *membrane* in *xy*-plane attached along boundary;
- Make it *vibrate*;
- Let $z(x,y,t)$ denote *displacement* of (x,y) at time t:

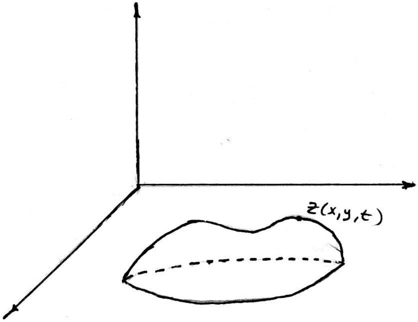
Hearing the Shape

- Consider a *membrane* in *xy*-plane attached along boundary;
- Make it *vibrate*;
- Let $z(x, y, t)$ denote *displacement* of (x, y) at time t:



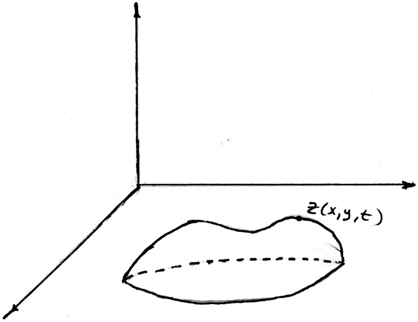
Hearing the Shape

- Consider a *membrane* in *xy*-plane attached along boundary;
- Make it *vibrate*;
- Let $z(x,y,t)$ denote *displacement* of (x,y) at time t:



Hearing the Shape

- Consider a *membrane* in *xy*-plane attached along boundary;
- Make it *vibrate*;
- Let $z(x, y, t)$ denote *displacement* of (x, y) at time t:



Wave equation and Laplacian

Then z satisfies *wave equation*:

$$\frac{\partial^2 z}{\partial t^2} - c^2 \Delta z = 0, \qquad z \equiv 0 \ \text{ on boundary}$$

Wave equation and Laplacian

Then z satisfies *wave equation*:

$$\frac{\partial^2 z}{\partial t^2} - c^2 \Delta z = 0, \qquad z \equiv 0 \ \text{ on } \ \text{boundary}$$

where

$$\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

is the Laplacian.

Wave equation and Laplacian

Then z satisfies *wave equation*:

$$\frac{\partial^2 z}{\partial t^2} - c^2 \Delta z = 0, \qquad z \equiv 0 \text{ on boundary}$$

where

$$\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

is the Laplacian.

(For simplicity, we take $c = 1$ in the sequel.)

Harmonics

We are particularly interested in *solutions of the form*:

$$z(x,y,t) = u(x,y)e^{i\omega t}, \qquad \omega = \text{frequency}.$$

Harmonics

We are particularly interested in *solutions of the form*:

$$z(x,y,t) = u(x,y)e^{i\omega t}, \qquad \omega = \text{frequency.}$$

Then all points of membrane oscillate with same frequency **but** possibly with different amplitudes.

Harmonics

We are particularly interested in *solutions of the form*:

$$z(x,y,t) = u(x,y)e^{i\omega t}, \qquad \omega = \text{frequency}.$$

Then all points of membrane oscillate with same frequency **but** possibly with different amplitudes.

- **For a musician:** membrane produces a *harmonic* sound of frequency ω

Harmonics

We are particularly interested in *solutions of the form*:

$$z(x,y,t) = u(x,y)e^{i\omega t}, \qquad \omega = \text{frequency}.$$

Then all points of membrane oscillate with same frequency **but** possibly with different amplitudes.

- **For a musician:** membrane produces a *harmonic* sound of frequency ω

(**then** ω is one of *overtones* (or *harmonics*) of membrane.)

Eigenvalues of Laplacian

- **For a mathematician:** these solutions are the ones furnished by *Fourier's method of separation of variables*.

Eigenvalues of Laplacian

- **For a mathematician:** these solutions are the ones furnished by *Fourier's method of separation of variables*.

  (**then** any solution is an infinite sum of these *special solutions*.)

Eigenvalues of Laplacian

- **For a mathematician:** these solutions are the ones furnished by *Fourier's method of separation of variables*.
  (**then** any solution is an infinite sum of these *special solutions*.)

IMPORTANT: ω cannot be arbitrary.

Eigenvalues of Laplacian

- **For a mathematician:** these solutions are the ones furnished by *Fourier's method of separation of variables*.
  (**then** any solution is an infinite sum of these *special solutions*.)

IMPORTANT: ω cannot be arbitrary.

Indeed, substituting we obtain

$$(-\Delta)u = \omega^2 u \quad \text{(and } u = 0 \text{ on the } \textit{boundary}\text{)}$$

Eigenvalues of Laplacian

- **For a mathematician:** these solutions are the ones furnished by *Fourier's method of separation of variables*.
  (**then** any solution is an infinite sum of these *special solutions*.)

IMPORTANT: ω cannot be arbitrary.

Indeed, substituting we obtain

$$(-\Delta)u = \omega^2 u \quad \text{(and } u = 0 \text{ on the } \textit{boundary}\text{)}$$

Thus, *harmonics* in sound produced by membrane have to do with

eigenvalues of Laplacian.

So, *hearing the shape* = ability to recover a geometric object from spectral data for corresponding Laplacian.

So, *hearing the shape* = ability to recover a geometric object from spectral data for corresponding Laplacian.

Note spectral data **must** include *multiplicities*.

So, *hearing the shape* = ability to recover a geometric object from spectral data for corresponding Laplacian.

Note spectral data **must** include *multiplicities*.

WHY DO WE EXPECT TO BE ABLE TO "HEAR THE SHAPE"?

1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity

2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences

3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

Here $\Delta = \dfrac{d^2}{dx^2}$,

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0.$

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0.$

- For each $\lambda > 0$, there are two linearly independent solutions

$$\sin\left(\sqrt{\lambda}x\right) \text{ and } \cos\left(\sqrt{\lambda}x\right).$$

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0.$

- For each $\lambda > 0$, there are two linearly independent solutions

$$\sin\left(\sqrt{\lambda}x\right) \quad \text{and} \quad \cos\left(\sqrt{\lambda}x\right).$$

- Assume that u describes profile of vibrating string of length ℓ with attached ends

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0.$

- For each $\lambda > 0$, there are two linearly independent solutions

$$\sin\left(\sqrt{\lambda}x\right) \text{ and } \cos\left(\sqrt{\lambda}x\right).$$

- Assume that u describes profile of vibrating string of length ℓ with attached ends $\Rightarrow$ $u(0) = u(\ell) = 0.$

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0$.

- For each $\lambda > 0$, there are two linearly independent solutions

$$\sin\left(\sqrt{\lambda}x\right) \quad \text{and} \quad \cos\left(\sqrt{\lambda}x\right).$$

- Assume that u describes profile of vibrating string of length ℓ with attached ends $\Rightarrow$ $u(0) = u(\ell) = 0$.

With locally symmetric spaces in mind, we instead consider condition $u(x + \ell) = u(x)$, i.e. u is defined on $\mathbb{R}/\ell\mathbb{Z} \simeq$ *circumference* of length ℓ.

Here $\Delta = \dfrac{d^2}{dx^2}$, so $\dfrac{d^2u}{dx^2} + \lambda u = 0.$

- For each $\lambda > 0$, there are two linearly independent solutions

$$\sin\left(\sqrt{\lambda}x\right) \text{ and } \cos\left(\sqrt{\lambda}x\right).$$

- Assume that u describes profile of vibrating string of length ℓ with attached ends $\Rightarrow$ $u(0) = u(\ell) = 0.$

With locally symmetric spaces in mind, we instead consider condition $u(x+\ell) = u(x)$, i.e. u is defined on $\mathbb{R}/\ell\mathbb{Z} \simeq$ *circumference* of length ℓ.

(Both conditions result in same *eigenvalues*.)

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

for which there are solutions with period ℓ.

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

for which there are solutions with period ℓ.

Each λ_n $(n > 0)$ has *multiplicity* 2 with eigenfunctions

$$\sin\left(\frac{2\pi n x}{\ell}\right) \quad \text{and} \quad \cos\left(\frac{2\pi n x}{\ell}\right).$$

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

for which there are solutions with period ℓ.

Each λ_n $(n > 0)$ has *multiplicity* 2 with eigenfunctions

$$\sin\left(\frac{2\pi n x}{\ell}\right) \quad \text{and} \quad \cos\left(\frac{2\pi n x}{\ell}\right).$$

- The sequence $\{\lambda_n\}$ determines ℓ.

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

for which there are solutions with period ℓ.

Each λ_n $(n > 0)$ has *multiplicity* 2 with eigenfunctions

$$\sin\left(\frac{2\pi n x}{\ell}\right) \quad \text{and} \quad \cos\left(\frac{2\pi n x}{\ell}\right).$$

- The sequence $\{\lambda_n\}$ determines ℓ.
- **So**, one **can** hear shape of circumference (or string)

Then $\sqrt{\lambda}\ell = 2\pi n$ for $n \in \mathbb{Z}$ $\Rightarrow$ we obtain a discrete sequence of eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\ell^2}, \quad n = 0, 1, \ldots$$

for which there are solutions with period ℓ.

Each λ_n $(n > 0)$ has *multiplicity* 2 with eigenfunctions

$$\sin\left(\frac{2\pi n x}{\ell}\right) \quad \text{and} \quad \cos\left(\frac{2\pi n x}{\ell}\right).$$

- The sequence $\{\lambda_n\}$ determines ℓ.
- **So**, one **can** hear shape of circumference (or string) (even if one misses a couple of low overtones).

1 Eigenvalue rigidity and hearing the shape of a drum
- Classical vs. Eigenvalue Rigidity

2 Hearing the Shape
- 1-dimensional case
- Flat tori of dimension > 1
- Weyl's Law and its Consequences

3 Locally symmetric spaces
- Laplace-Beltrami operator
- Isospectral non-isometric manifolds
- Our results

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

Here *eigenfunctions* of Laplacian are

$$\sin(2\pi mx)\cos(2\pi ny), \quad \sin(2\pi mx)\cos(2\pi nx),$$

$$\text{and} \quad \cos(2\pi mx)\cos(2\pi ny) \quad (m, n \in \mathbb{Z}).$$

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

Here *eigenfunctions* of Laplacian are

$$\sin(2\pi mx)\cos(2\pi ny), \quad \sin(2\pi mx)\cos(2\pi nx),$$

$$\text{and} \quad \cos(2\pi mx)\cos(2\pi ny) \quad (m, n \in \mathbb{Z}).$$

Corresponding *eigenvalue* $\lambda_{m,n} = 4\pi^2(m^2 + n^2)$.

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

Here *eigenfunctions* of Laplacian are

$$\sin(2\pi mx)\cos(2\pi ny), \quad \sin(2\pi mx)\cos(2\pi nx),$$

$$\text{and} \quad \cos(2\pi mx)\cos(2\pi ny) \quad (m, n \in \mathbb{Z}).$$

Corresponding *eigenvalue* $\lambda_{m,n} = 4\pi^2(m^2 + n^2)$.

This *spectral data* can be packaged differently:

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

Here *eigenfunctions* of Laplacian are

$$\sin(2\pi mx)\cos(2\pi ny), \quad \sin(2\pi mx)\cos(2\pi nx),$$

$$\text{and} \quad \cos(2\pi mx)\cos(2\pi ny) \quad (m, n \in \mathbb{Z}).$$

Corresponding *eigenvalue* $\lambda_{m,n} = 4\pi^2(m^2 + n^2)$.

This *spectral data* can be packaged differently:

- eigenfunctions are obtained from

$$\sin(2\pi \cdot (m,n) \cdot (x,y)) \quad \text{and} \quad \cos(2\pi \cdot (m,n) \cdot (x,y));$$

2-dimensional torus

Consider "canonical" 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

Here *eigenfunctions* of Laplacian are

$$\sin(2\pi mx)\cos(2\pi ny), \quad \sin(2\pi mx)\cos(2\pi nx),$$

$$\text{and} \quad \cos(2\pi mx)\cos(2\pi ny) \quad (m, n \in \mathbb{Z}).$$

Corresponding *eigenvalue* $\lambda_{m,n} = 4\pi^2(m^2 + n^2)$.

This *spectral data* can be packaged differently:

- eigenfunctions are obtained from

$$\sin(2\pi \cdot (m,n) \cdot (x,y)) \quad \text{and} \quad \cos(2\pi \cdot (m,n) \cdot (x,y));$$

- eigenvalue $= 4\pi^2 q(m,n)$, where $q(a,b) = a^2 + b^2$.

d-dimensional torus

Generalization to an arbitrary torus $\mathbb{R}^d/L$:

(where L is a lattice of rank d)

d-dimensional torus

Generalization to an arbitrary torus $\mathbb{R}^d/L$:

(where L is a lattice of rank d)

- eigenfunctions are obtained from

$$\sin(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x}) \quad \text{and} \quad \cos(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x})$$

where $\overrightarrow{x} = (x_1, \ldots, x_d)$ and $\overrightarrow{m} = (m_1, \ldots, m_d) \in L^*$ (*dual* lattice);

d-dimensional torus

Generalization to an arbitrary torus $\mathbb{R}^d/L$:
(where L is a lattice of rank d)

- eigenfunctions are obtained from

$$\sin(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x}) \quad \text{and} \quad \cos(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x})$$

where $\overrightarrow{x} = (x_1, \ldots, x_d)$ and $\overrightarrow{m} = (m_1, \ldots, m_d) \in L^*$ (*dual* lattice);

- eigenvalue = $4\pi^2 q(\overrightarrow{m})$ where q is the standard quadratic form on $\mathbb{R}^d$

d-dimensional torus

Generalization to an arbitrary torus $\mathbb{R}^d/L$:

(where L is a lattice of rank d)

- eigenfunctions are obtained from

$$\sin(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x}) \quad \text{and} \quad \cos(2\pi \cdot \overrightarrow{m} \cdot \overrightarrow{x})$$

where $\overrightarrow{x} = (x_1, \dots, x_d)$ and $\overrightarrow{m} = (m_1, \dots, m_d) \in L^*$ (*dual* lattice);

- eigenvalue $= 4\pi^2 q(\overrightarrow{m})$ where q is the standard quadratic form on $\mathbb{R}^d$

Recall that $L^* := \{\overrightarrow{x} \in \mathbb{R}^d \mid \overrightarrow{x} \cdot \overrightarrow{y} \in \mathbb{Z}$ for all $\overrightarrow{y} \in L\}$.

d-dimensional torus

So, two tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 :

d-dimensional torus

So, two tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 :

- have same eigenvalues of Laplacian $\Leftrightarrow$ $q_1(L_1^*) = q_2(L_2^*)$

d-dimensional torus

So, two tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 :

- have same eigenvalues of Laplacian $\Leftrightarrow$ $q_1(L_1^*) = q_2(L_2^*)$
- are isometric $\Leftrightarrow$ $d_1 = d_2 =: d$, hence $q_1 = q_2 =: q$, **and** $\exists\ \sigma \in O_d(q)$ such that $\sigma(L_1) = L_2$ (equivalently, $\sigma(L_1^*) = L_2^*$)

d-dimensional torus

So, two tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 :

- have same eigenvalues of Laplacian $\Leftrightarrow$ $q_1(L_1^*) = q_2(L_2^*)$
- are isometric $\Leftrightarrow$ $d_1 = d_2 =: d$, hence $q_1 = q_2 =: q$, **and** $\exists\ \sigma \in O_d(q)$ such that $\sigma(L_1) = L_2$ (equivalently, $\sigma(L_1^*) = L_2^*$)

(in other words, quadratic lattices $(L_1, q_1|L_1)$ and $(L_2, q_2|L_2)$ are isomorphic)

d-dimensional torus

So, two tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 :

- have same eigenvalues of Laplacian $\Leftrightarrow$ $q_1(L_1^*) = q_2(L_2^*)$
- are isometric $\Leftrightarrow$ $d_1 = d_2 =: d$, hence $q_1 = q_2 =: q$, **and** $\exists\ \sigma \in O_d(q)$ such that $\sigma(L_1) = L_2$ (equivalently, $\sigma(L_1^*) = L_2^*$)

(in other words, quadratic lattices $(L_1, q_1|L_1)$ and $(L_2, q_2|L_2)$ are isomorphic)

Can see strong *arithmetic connection* foreshadowing the use of

arithmetic groups.

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

This yields nonisometric tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 that have same eigenvalues of Laplacian.

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

This yields nonisometric tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 that have same eigenvalues of Laplacian.

Moreover, one can construct such tori

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

This yields nonisometric tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 that have same eigenvalues of Laplacian.

Moreover, one can construct such tori

- having different dimensions;

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

This yields nonisometric tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 that have same eigenvalues of Laplacian.

Moreover, one can construct such tori

- having different dimensions;
- having same dimension, **but** different volumes.

d-dimensional torus

But it is very easy to construct *positive definite integral* quadratic forms q_1 and q_2 that

- represents same integers, **but**
- are **not** equivalent even over $\mathbb{Q}$.

This yields nonisometric tori $\mathbb{R}^{d_1}/L_1$ and R^{d_2}/L_2 that have same eigenvalues of Laplacian.

Moreover, one can construct such tori

- having different dimensions;
- having same dimension, **but** different volumes.

(In fact, there are *infinite families* of such tori having same dimension **but** pairwise different volumes.)

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

This makes Kac's question somewhat rhetorical as it is unclear how we can possibly hear multiplicities, BUT ...

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

This makes Kac's question somewhat rhetorical as it is unclear how we can possibly hear multiplicities, BUT ...

Definition.

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

This makes Kac's question somewhat rhetorical as it is unclear how we can possibly hear multiplicities, BUT ...

Definition.

- *Laplace spectrum* = set of eigenvalues of Laplacian **with** multiplicities

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

This makes Kac's question somewhat rhetorical as it is unclear how we can possibly hear multiplicities, BUT ...

Definition.

- *Laplace spectrum* = set of eigenvalues of Laplacian **with** multiplicities (assuming that these are finite)

So, expectation to *hear the shape* fails **if** one considers only *eigenvalues of Laplacian*.

The way to fix situation is to consider

eigenvalues *with multiplicities*.

This makes Kac's question somewhat rhetorical as it is unclear how we can possibly hear multiplicities, BUT ...

Definition.

- *Laplace spectrum* = set of eigenvalues of Laplacian **with** multiplicities (assuming that these are finite)
- Two geometric objects are *isospectral* **if** they have same Laplace spectra.

1 Eigenvalue rigidity and hearing the shape of a drum
- Classical vs. Eigenvalue Rigidity

2 Hearing the Shape
- 1-dimensional case
- Flat tori of dimension > 1
- Weyl's Law and its Consequences

3 Locally symmetric spaces
- Laplace-Beltrami operator
- Isospectral non-isometric manifolds
- Our results

Weyl's Law

In 1911, Hermann Weyl established a particular case of

Weyl's Law.

Weyl's Law

In 1911, Hermann Weyl established a particular case of

Weyl's Law.

Let M be a compact Riemannian manifold of dimension d,

Weyl's Law

In 1911, Hermann Weyl established a particular case of

Weyl's Law.

Let M be a compact Riemannian manifold of dimension d, and let

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

be sequence of eigenvalues of Laplacian with multiplicities.

Weyl's Law

In 1911, Hermann Weyl established a particular case of

Weyl's Law.

Let M be a compact Riemannian manifold of dimension d, and let

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

be sequence of eigenvalues of Laplacian with multiplicities.

For $\lambda > 0$, set $N(\lambda) = \#\{ j \mid \sqrt{\lambda_j} \leqslant \lambda \}$.

Weyl's Law

In 1911, Hermann Weyl established a particular case of

Weyl's Law.

Let M be a compact Riemannian manifold of dimension d, and let

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

be sequence of eigenvalues of Laplacian with multiplicities.

For $\lambda > 0$, set $N(\lambda) = \#\{ j \mid \sqrt{\lambda_j} \leqslant \lambda \}$.

Then

$$N(\lambda) = \frac{\mathrm{vol}(M)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^d + o(\lambda^d)$$

Consequently, isospectral compact Riemannian manifolds have

Consequently, isospectral compact Riemannian manifolds have

- same *dimension;*

Consequently, isospectral compact Riemannian manifolds have

- same *dimension;*
- same *volume.*

Consequently, isospectral compact Riemannian manifolds have

- same *dimension;*
- same *volume.*

So, if $\mathbb{R}^{d_1}/L_1$ and $\mathbb{R}^{d_2}/L_2$ are isospectral, **then:**

Consequently, isospectral compact Riemannian manifolds have

- same *dimension;*
- same *volume.*

So, if $\mathbb{R}^{d_1}/L_1$ and $\mathbb{R}^{d_2}/L_2$ are isospectral, **then:**

- $d_1 = d_2$;

Consequently, isospectral compact Riemannian manifolds have

- same *dimension;*
- same *volume.*

So, if $\mathbb{R}^{d_1}/L_1$ and $\mathbb{R}^{d_2}/L_2$ are isospectral, **then:**

- $d_1 = d_2$;
- corresponding quadratic forms q_1 and q_2 have same *discriminant*.

GAUSS: There are finitely many equivalence classes of positive definite integral quadratic forms of a given dimension and a given discriminant.

GAUSS: There are finitely many equivalence classes of positive definite integral quadratic forms of a given dimension and a given discriminant.

- Assume that quadratic form has *integral values* on L.

GAUSS: There are finitely many equivalence classes of positive definite integral quadratic forms of a given dimension and a given discriminant.

- Assume that quadratic form has *integral values* on L. **Then** there are finitely many $\mathbb{R}^d/M$ isospectral to $\mathbb{R}^d/L$.

GAUSS: There are finitely many equivalence classes of positive definite integral quadratic forms of a given dimension and a given discriminant.

- Assume that quadratic form has *integral values* on L. **Then** there are finitely many $\mathbb{R}^d/M$ isospectral to $\mathbb{R}^d/L$.

- M. KNESER: *Finiteness* holds for any lattice (not necessarily integral-valued).

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

By J. Milnor

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

Bad news

In 1964, MILNOR re-discovered Witt's paper (1941)

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, i x \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \epsilon R^n$ such that $x \cdot y$ is an integer for all $x \epsilon L$. Then each $y \epsilon L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, MILNOR re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, MILNOR re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer $\Rightarrow$

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, MILNOR re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer $\Rightarrow$

- non-isometric isospectral 16-dim tori.

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, MILNOR re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer $\Rightarrow$

- non-isometric isospectral 16-dim tori.
- $\dim = 12$ Kneser

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

By J. Milnor

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, Milnor re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer $\Rightarrow$

- non-isometric isospectral 16-dim tori.
- $\dim = 12$ Kneser
- $\dim = 8$ Kitaoka

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

In 1964, MILNOR re-discovered Witt's paper (1941) containing two noniquivalent unimodular integral quadratic forms $\dim = 16$ with same number of representations of each integer $\Rightarrow$

- non-isometric isospectral 16-dim tori.

- $\dim = 12$ Kneser
- $\dim = 8$ Kitaoka
- $\dim = 4$ Schiemann, Conway-Sloane

1 Eigenvalue rigidity and hearing the shape of a drum
- Classical vs. Eigenvalue Rigidity

2 Hearing the Shape
- 1-dimensional case
- Flat tori of dimension > 1
- Weyl's Law and its Consequences

3 Locally symmetric spaces
- Laplace-Beltrami operator
- Isospectral non-isometric manifolds
- Our results

1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity

2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences

3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

Let M be a Riemannian d-manifold.

Let M be a Riemannian d-manifold.

Laplace - Beltrami operator in local coordinates $x_1, \ldots, x_d$:

$$\Delta(u) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \, \frac{\partial u}{\partial x_j} \right), \qquad \text{(L)}$$

where $g = (g_{ij})$ is matrix of Riemannian metric, $g^{-1} = (g^{ij})$.

Let M be a Riemannian d-manifold.

Laplace - Beltrami operator in local coordinates $x_1, \ldots, x_d$:

$$\Delta(u) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\,\frac{\partial u}{\partial x_j}\right), \qquad \text{(L)}$$

where $g = (g_{ij})$ is matrix of Riemannian metric, $g^{-1} = (g^{ij})$.

Example.

For upper-half plane $\mathbb{H} = \{x + iy \mid y > 0\}$, $ds^2 = y^{-2}(dx^2 + dy^2)$, we have

$$\Delta_{\mathbb{H}} = y^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;
- Δ is self-adjoint and *negative* definite
- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

For M compact:

- Δ is self-adjoint and *negative* definite

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

For M compact:

- Δ is self-adjoint and *negative* definite

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

For M compact:

- Δ is self-adjoint and *negative* definite
  (**so**, we add $-$ in (L) to make Δ *positive* definite - **then**)

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

For M compact:

- Δ is self-adjoint and *negative* definite
  (**so**, we add $-$ in (L) to make Δ *positive* definite - **then**)

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:
  $$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

For M compact:

- Δ is self-adjoint and *negative* definite
  (**so**, we add $-$ in (L) to make Δ *positive* definite - **then**)

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:
  $$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

However, eigenvalues are extremely hard to compute!

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are

$$\geqslant 1/4.$$

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are $\geqslant 1/4$.

Selberg proved estimation $\geqslant 3/16$.

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are

$$\geqslant 1/4.$$

Selberg proved estimation $\geqslant 3/16$.

Gap of $1/16$ has not been bridged in over 50 years!

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are

$$\geqslant 1/4.$$

Selberg proved estimation $\geqslant 3/16$.

Gap of $1/16$ has not been bridged in over 50 years!

Best known estimate $\geqslant \dfrac{171}{784} = 0.218\ldots$ (Luo–Rudnick–Sarnak, 1995)

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are

$$\geqslant 1/4.$$

Selberg proved estimation $\geqslant 3/16$.

Gap of $1/16$ has not been bridged in over 50 years!

Best known estimate $\geqslant \dfrac{171}{784} = 0.218\ldots$ (Luo–Rudnick–Sarnak, 1995)

We will describe techniques to *bypass* explicit computations!

1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity

2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences

3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

- M.-F. Vigneras (1980) - first example of isospectral non-isometric *Riemann surfaces*.

- M.-F. Vigneras (1980) - first example of isospectral non-isometric *Riemann surfaces*.

Construction used <u>arithmetic</u> of quaternion algebras.

- M.-F. Vigneras (1980) - first example of isospectral non-isometric *Riemann surfaces*.

Construction used arithmetic of quaternion algebras.

- T. Sunada (1985) gave a *different* construction, based on group-theoretic properties of fundamental group.

- M.-F. Vigneras (1980) - first example of isospectral non-isometric *Riemann surfaces*.

Construction used arithmetic of quaternion algebras.

- T. Sunada (1985) gave a *different* construction, based on group-theoretic properties of fundamental group.

Can be implemented for *general* locally symmetric spaces.

- M.-F. Vigneras (1980) - first example of isospectral non-isometric *Riemann surfaces*.

Construction used arithmetic of quaternion algebras.

- T. Sunada (1985) gave a *different* construction, based on group-theoretic properties of fundamental group.

Can be implemented for *general* locally symmetric spaces.

BOTH constructions result in commensurable manifolds.

Two Riemannian manifolds M_1 and M_2 are *commensurable*

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ M_1 & & M_2 \end{tikzcd}$$

"Right question": *Are two isospectral (compact Riemannian) manifolds necessarily commensurable?*

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

"Right question": *Are two isospectral (compact Riemannian) manifolds necessarily commensurable?*

- In general, **no** - Lubotzky et al. (2006)

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$
\begin{array}{ccc}
 & M & \\
\swarrow & & \searrow \\
M_1 & & M_2
\end{array}
$$

"Right question": *Are two isospectral (compact Riemannian) manifolds necessarily commensurable?*

- In general, **no** - Lubotzky et al. (2006)
- Prasad-R.: **yes** in many cases

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$
\begin{array}{ccc}
 & M & \\
\swarrow & & \searrow \\
M_1 & & M_2
\end{array}
$$

"Right question": *Are two isospectral (compact Riemannian) manifolds necessarily commensurable?*

- In general, **no** - Lubotzky et al. (2006)
- Prasad-R.: **yes** in many cases (Publ. math. IHES **109**(2009))

Two Riemannian manifolds M_1 and M_2 are *commensurable* **if** they have a common finite-sheeted cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

"Right question": *Are two isospectral (compact Riemannian) manifolds necessarily commensurable?*

- In general, **no** - Lubotzky et al. (2006)
- Prasad-R.: **yes** in many cases (Publ. math. IHES **109**(2009))

Previously, results were available only for hyperbolic 2- and 3-manifolds. (A. Reid et al.)

1 Eigenvalue rigidity and hearing the shape of a drum
- Classical vs. Eigenvalue Rigidity

2 Hearing the Shape
- 1-dimensional case
- Flat tori of dimension > 1
- Weyl's Law and its Consequences

3 Locally symmetric spaces
- Laplace-Beltrami operator
- Isospectral non-isometric manifolds
- Our results

Notations

Notations

Let G be a semi-simple algebraic $\mathbb{R}$-group; $\mathcal{G} = G(\mathbb{R})$.

Notations

Let G be a semi-simple algebraic $\mathbb{R}$-group; $\mathcal{G} = G(\mathbb{R})$.

- $\mathcal{K}$ - maximal compact subgroup of $\mathcal{G}$;

  $\mathfrak{X} := \mathcal{K}\backslash\mathcal{G}$ - corresponding symmetric space.

Notations

Let G be a semi-simple algebraic $\mathbb{R}$-group; $\mathcal{G} = G(\mathbb{R})$.

- $\mathcal{K}$ - maximal compact subgroup of $\mathcal{G}$;
  $\mathfrak{X} := \mathcal{K}\backslash\mathcal{G}$ - corresponding symmetric space.

- For $\Gamma \subset \mathcal{G}$ discrete torsion free subgroup,
  $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ - corresponding locally symmetric space.

Notations

Let G be a semi-simple algebraic $\mathbb{R}$-group; $\mathcal{G} = G(\mathbb{R})$.

- $\mathcal{K}$ - maximal compact subgroup of $\mathcal{G}$;

  $\mathfrak{X} := \mathcal{K}\backslash\mathcal{G}$ - corresponding symmetric space.

- For $\Gamma \subset \mathcal{G}$ discrete torsion free subgroup,

  $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ - corresponding locally symmetric space.

- $\mathfrak{X}_\Gamma$ is *arithmetically defined* if Γ is *arithmetic*.

Notations

Let G be a semi-simple algebraic $\mathbb{R}$-group; $\mathcal{G} = G(\mathbb{R})$.

- $\mathcal{K}$ - maximal compact subgroup of $\mathcal{G}$;

  $\mathfrak{X} := \mathcal{K}\backslash\mathcal{G}$ - corresponding symmetric space.

- For $\Gamma \subset \mathcal{G}$ discrete torsion free subgroup,

  $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ - corresponding locally symmetric space.
- $\mathfrak{X}_\Gamma$ is *arithmetically defined* if Γ is *arithmetic*.

Now, let G_1 and G_2 be *absolutely almost simple* $\mathbb{R}$-groups,

$\Gamma_i \subset \mathcal{G}_i = G_i(\mathbb{R})$ be a discrete torsion-free subgroup,

$\mathfrak{X}_{\Gamma_i}$ - corresponding locally symmetric space, $i = 1, 2$.

Theorem 1 (Prasad-R.)

Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be compact locally symmetric spaces, and assume that they are isospectral.

Theorem 1 (Prasad-R.)

Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be compact locally symmetric spaces, and assume that they are isospectral.

(1) *If $\mathfrak{X}_{\Gamma_1}$ is arithmetically defined, then so is $\mathfrak{X}_{\Gamma_2}$.*

Theorem 1 (Prasad-R.)

Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be compact locally symmetric spaces, and assume that they are isospectral.

(1) *If $\mathfrak{X}_{\Gamma_1}$ is arithmetically defined, then so is $\mathfrak{X}_{\Gamma_2}$.*

(2) $G_1 = G_2 =: G$, *hence $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ have same universal cover.*

Theorem 1 (Prasad-R.)

Let $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be compact locally symmetric spaces, and* *assume* *that they are* *isospectral.*

(1) *If* $\mathfrak{X}_{\Gamma_1}$ *is arithmetically defined, then so is* $\mathfrak{X}_{\Gamma_2}$.

(2) $G_1 = G_2 =: G$, *hence* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *have same universal cover.*

(3) *If at least one of the groups* Γ_1 *or* Γ_2 *is* *arithmetic*, *then unless* G *is of type* A_n $(n > 1)$, D_{2n+1} $(n > 1)$ *or* E_6, *spaces* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

Geometric applications

Corollary

Let M_1 and M_2 be arithmetically defined hyperbolic manifolds of dimension $d \not\equiv 1 \pmod 4$. If M_1 and M_2 are isospectral then they are commensurable.

Geometric applications

Corollary

Let M_1 *and* M_2 *be arithmetically defined hyperbolic manifolds of dimension* $d \not\equiv 1 (\mathrm{mod}\, 4)$. *If* M_1 *and* M_2 *are isospectral then they are commensurable.*

Our techniques apply to locally symmetric spaces that share a different set of geometric data, viz. *length spectrum*.

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

$\mathbb{Q} \cdot L(M)$ is rational length spectrum.

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

$\mathbb{Q} \cdot L(M)$ is rational length spectrum.

It has *less* geometric content,

Definition.

(1) Let M be a Riemannian manifold.

Length spectrum $L(M)$ = set of length of all closed geodesics.

(2) Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

$\mathbb{Q} \cdot L(M)$ is rational length spectrum.

It has *less* geometric content, **but** may be *easier* to figure out, and it is invariant under passing to a commensurable manifold.

Using *trace formula* one shows that

Using *trace formula* one shows that

compact $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ isospectral $\Rightarrow$ $L(M_1) = L(M_2)$

Using *trace formula* one shows that

compact $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ isospectral $\Rightarrow$ $L(M_1) = L(M_2)$

$$\Rightarrow \quad \mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2).$$

Using *trace formula* one shows that

$$\text{compact } \mathfrak{X}_{\Gamma_1} \text{ and } \mathfrak{X}_{\Gamma_2} \text{ isospectral } \Rightarrow L(M_1) = L(M_2)$$
$$\Rightarrow \mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2).$$

Most of our geometric results rely only on assumption that locally symmetric spaces are *length-commensurable*.

Using *trace formula* one shows that

$$\text{compact } \mathfrak{X}_{\Gamma_1} \text{ and } \mathfrak{X}_{\Gamma_2} \text{ isospectral } \Rightarrow L(M_1) = L(M_2)$$
$$\Rightarrow \mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2).$$

Most of our geometric results rely only on assumption that locally symmetric spaces are *length-commensurable*.

Length-commensurability is translated into *weak commensurability* of fundamental groups.