HOMEWORK #2 SOLUTIONS

(1) $$\frac{3\left(\frac{88!}{28!\,30!\,30!}\right)}{\frac{90!}{30!\,30!\,30!}} = \frac{3(30)(29)}{(89)(90)} = \frac{29}{89}$$

(2) a) $\dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$ b) $\dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$

(3) a) $$P(\text{Klingon} \mid \text{voted}) = \frac{P(\text{Klingon} \cap \text{voted})}{P(\text{voted})} = \frac{P(\text{voted} \mid \text{Klingon})\,P(\text{Klingon})}{P(\text{voted} \cap \text{Klingon}) + P(\text{voted} \cap \text{Romulan}) + P(\text{voted} \cap \text{Vulcan}}$$

$$= \frac{.10\,(.45)}{.10\,(.45) + .5(.25) + 1(.30)} = \frac{.045}{.045 + .125 + .30} = \frac{.045}{.47} \approx .096$$

b) $$P(\text{Romulan} \mid \text{voted}) = \frac{.5(.25)}{.10(.45) + .5(.25) + 1(.30)} = \frac{.125}{.47} \approx .266$$

c) .47

(4) Let E be the event that the ball chosen from urn A was white

Let F be the event that exactly 2 white balls were selected.

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{\frac{2}{7}\left(\frac{7}{10}\right)\left(\frac{3}{4}\right) + \frac{2}{7}\left(\frac{3}{10}\right)\left(\frac{1}{4}\right)}{\frac{5}{7}\left(\frac{7}{10}\right)\left(\frac{1}{4}\right) + \frac{2}{7}\left(\frac{7}{10}\right)\left(\frac{3}{4}\right) + \frac{2}{7}\left(\frac{3}{10}\right)\left(\frac{1}{4}\right)} = \frac{\frac{3}{20} + \frac{3}{140}}{\frac{1}{8} + \frac{3}{20} + \frac{3}{140}} = \frac{\frac{48}{280}}{\frac{83}{280}} = \frac{48}{83}$$

(5) a) Axiom 1 says $P(A \cup B) \le 1$. Substituting for $P(A \cup B)$, we get $P(A) + P(B) - P(AB) \le 1$. Thus $P(AB) \ge P(A) + P(B) - 1$.

b) By part a), $P(ABC) \ge P(AB) + P(C) - 1$. By part a) again, this is $\ge \big(P(A) + P(B) - 1\big) + P(C) - 1$ which is $P(A) + P(B) + P(C) - 2$.

(6) a) $P(\text{different}) = \dfrac{30}{36} = \dfrac{5}{6}$

b) $$P(\text{larger} = 5 \mid \text{different}) = \frac{P(\{(1,5),(5,1),(2,5)(5,2),(5,3),(3,5),(5,4)(4,5)\})}{P(\text{different})}$$

$$= \frac{\frac{8}{36}}{\frac{30}{36}} = \frac{4}{15}$$

c) $P(\text{sum is even} \mid \text{different}) = \dfrac{P(\text{sum is even and numbers different})}{P(\text{different})}$

$$= \frac{\frac{12}{36}}{30/36} = \frac{12}{30} = \frac{2}{5}$$

d) $P(\text{different} \mid \text{sum is even}) = \dfrac{\frac{12}{36}}{\frac{18}{36}} = \dfrac{12}{18} = \dfrac{2}{3}$

7) Assume urn A and urn B each has n balls. At each of the 4 interchanges, a ball from urn A and a ball from urn B are chosen, so there are n^8 total ways the experiment can be performed. On the second interchange, we either return ~~both~~ balls interchanged on the first interchange (event E_1), or return only one of the two interchanged on the first interchange (event E_2), or return neither (event E_3). Then the event that all balls return to where they started is $E_1 \cup E_2 \cup E_3$, and the probability is

$$\frac{n^4 + 4n^2(n-1)^2 + n^2(n-1)^2(2)}{n^8} = \frac{8(n-1)^2 + n^2}{n^6}$$

(when $n=1$, this is 1. When $n=2$, this is $\frac{12}{64}$)

8) $P(\text{white from remaining 3} \mid \text{black drawn}) = \dfrac{\frac{4}{9}\left(\frac{5}{8}\right)}{\frac{4}{9}} = \dfrac{5}{8}$

9) a) $P(\text{two-tailed} \mid \text{shows } T) = \dfrac{P(\text{two-tailed} \cap \text{shows } T)}{P(\text{shows } T)}$

$$= \frac{P(\text{shows } T \mid \text{two-tailed})P(\text{two-tailed})}{P(\text{shows } T \mid \text{two-tailed})P(\text{two-tailed}) + P(\text{shows } T \mid \text{fair})P(\text{fair})}$$

$$= \frac{1(.5)}{1(.5) + \frac{1}{2}\left(\frac{1}{2}\right)} = \frac{2}{3}$$

b) $P(\text{two-tailed} \mid \text{shows T three times})$

$$= \frac{1(.5)}{1(.5) + (.5)^3(.5)} = \frac{.5}{\frac{9}{16}} = \frac{8}{9}$$

(9) c) For general n, the probability is $\dfrac{1(\frac{1}{2})}{1(\frac{1}{2}) + (\frac{1}{2})^n(\frac{1}{2})} = \dfrac{1}{1+(\frac{1}{2})^n} \to 1$ as $n \to \infty$

(10) a) $\binom{10}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5 = \dfrac{252\,(32)}{3^{10}} \approx .137$

b) $\displaystyle\sum_{j=5}^{10} \binom{10}{j}\left(\frac{1}{3}\right)^j\left(\frac{2}{3}\right)^{10-j} \approx .213$

(11) a) $\dfrac{\binom{40}{5}\binom{20}{5}}{\binom{60}{10}} \approx .135$

b) $\displaystyle\sum_{j=5}^{10} \frac{\binom{40}{10-j}\binom{20}{j}}{\binom{60}{10}} \approx .194$

(12) $P(\geq k$ show up$) = P(\geq k$ show up and the weather is good$)$

$+ P(\geq k$ show up and the weather is bad$)$

$= P(\geq k$ show up $|$ weather is good$)\,P($weather is good$)$

$+ P(\geq k$ show up $|$ weather is bad$)\,P($weather is bad$)$

$$= \sum_{i=k}^{n} \binom{n}{i} p_g^i (1-p_g)^{n-i} (1-q) + \sum_{i=k}^{n} \binom{n}{i} p_b^i (1-p_b)^{n-i} q$$

(13) This is a gambler's ruin problem. With the notation from page 96, $N = 80$, $i = 75$, $p = \frac{1}{3}$ $q = \frac{2}{3}$

$P(\text{Joe wins}) = \dfrac{1-2^{75}}{1-2^{80}} \approx \dfrac{1}{32}$