## Convexity and Combinatorics

### AMS Special Session

### March 7–8, 2015, Washington, DC

**Organizers:** Jim Lawrence and Valeriu Soltan, George Mason University, Virginia

### PROGRAM OF THE SESSION

**Saturday, March 7**

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| 8:00 am | W. Morris. *A Mihalisin–Klee theorem for fans.* |
| 8:30 am | G. Agnarsson. *General permutahedra.* |
| 9:00 am | C. Lee. *A generalization of the secondary polytope induced by lifting and deleting.* |
| 9:30 am | C. Toth. *Flip distances and Hamiltonian triangulations.* |
| 10:00am | B. Braun. *Unimodality problems in Ehrhart theory.* |
| 10:30am | W. Finbow-Singh. *Low dimensional neighbourly simplicial polytopes.* |
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| 2:00 pm | Fu Liu. *Ehrhart positivity for generalized permutohedra.* |
| 2:30 pm | E. Schulte. *Colorful polytopes, associahedra and cyclohedra.* |
| 3:00 pm | I. Scheidwasser. *Contractions of polygons in abstract polytopes.* |
| 3:30 pm | T. Bisztriczky. *Characterizations of cyclic polytopes.* |
| 4:00 pm | J. Lawrence. *Chains of antiprisms.* |

**Sunday, March 8**

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| 8:00 am | Z. Mustafaev. *Centered convex bodies and inequalities for cross-section measures.* |
| 8:30 am | G. Toth. *Dual mean Minkowski measures and the Grunbaum conjecture.* |
| 9:00 am | R. Dawson. *Some results on intangent spreads of convex bodies.* |
| 9:30 am | W. Kuperberg. *Approximating convex disks from inside and out by parallelograms.* |
| 10:00am | R. Howard. *Total diameter and area of closed submanifolds.* |
| 10:30am | D. Ismailescu. *Improved lower bounds for the chromatic number of several small dimensional Euclidean spaces.* |
| 11:00am | V. Soltan. *Convex hypersurfaces with hyperplanar intersections of their homothetic copies.* |

## ABSTRACTS

**Geir Agnarsson.** George Mason University, Fairfax, VA.

*General permutahedra.*

We first derive an exponential generating *flag function* of the standard permutahedron $\prod_{n-1}$. We then investigate the flags of a Minkowski sum of all the standard simplices $\Delta_{k-1}$ in $\mathbb{R}^n$ for fixed $k$ and $n$. These polytopes are all simple and naturally generalize both the standard simplex $\Delta_{n-1}$ and the permutahedron $\prod_{n-1}$ both of dimension $n-1$.

**András Bezdek, Wlodzimierz Kuperberg.*** Auburn University, Auburn, AL.

*Approximating convex disks from inside and out by parallelograms.*

For each convex disk $K$ we consider the minimum area $P(K)$ of a parallelogram containing $K$ and the maximum area $p(K)$ of a parallelogram contained in $K$, then we seek the maximum of $P(K)$ and the minimum of $p(K)$ over all convex disks $K$ of area 1. Without assuming central symmetry of $K$, the naturally anticipated answers will be given, but when we assume central symmetry, the problem of the maximum of $P(K)$ becomes much harder. We state a conjecture and discuss it in a quite broad context that includes the well-known, still unresolved Reinhardt Conjecture on the criticality of the smoothed octagon.

**Benjamin Braun.** University of Kentucky, Lexington, KY.

*Unimodality problems in Ehrhart theory.*

The $h^*$ polynomial of a convex lattice polytope $P$ is a subtle invariant related to geometric and algebraic properties of $P$. Unimodality of the coefficient sequence of the $h^*$ polynomial occurs often, yet the reasons for this are not well-understood. We will survey a selection of recent results regarding $h^*$-unimodality and discuss some interesting open problems in this area.

**Ted Bisztriczky.** University of Calgary, Calgary, Canada.

*Characterizations of cyclic polytopes.*

One of the most studied class of objects in geometry are the cyclic polytopes. The importance of theses polytopes is due to their usefulness not only to geometry and other branches of mathematics but also to other disciplines, such as economics and chemistry. We review some properties of a cyclic polytope and present some of its characterizations.

**Robert Dawson.** Saint Mary's University, Halifax, Canada.

*Some results on intangent spreads of convex bodies.*

By a spread of bodies we will understand a real $n$-manifold embedded in a (here metric) hyperspace – that is, an $n$-parameter family of bodies. Monotone spreads (those for which every two bodies are joined by a monotone arc) have been considered elsewhere (by the speaker) in the context of the Chebyshev nearest neighbor property. Here we consider a weaker property, that of intangency: a spread of bodies is intangent if no two elements have a common boundary point at which they share a supporting hyperplane (in the same sense). All monotone spreads have this property, as do many others. We will examine some implications of this property, which is sufficient to explain several observations about monotone spreads. We will see surprising (though elementary) connections to algebraic topology.

**Wendy Finbow-Singh.** Saint Mary's University, Halifax, Canada.

*Low dimensional neighbourly simplicial polytopes.*

Amongst the $d$-polytopes with $v$ vertices, the neighbourly polytopes have the greatest number of facets. This maximum property has prompted researchers to compose lists of them. In this talk, we will discuss an algorithm for generating the list of simplicial neighbourly $d$-polytopes with $v$ vertices, for a given dimension $d$ and number of vertices, $v$.

**Mohammad Ghomi, Ralph Howard.*** University of South Carolina, Columbia, SC.

*Total diameter and area of closed submanifolds.*

The *total diameter* of a closed planar curve $C$ is the integral its antipodal chord lengths. We show this quantity is bounded below by twice the area enclosed by the curve. Furthermore, when the curve is convex or centrally symmetric, the lower bound is twice as large. Both inequalities are sharp and equality holds in the convex case only when the curve is a circle. We generalize these results to the case of $m$-dimensional submanifolds of Euclidean $n$-space where the enclosed *area* is defined in terms of the mod 2 winding numbers of the submanifold about $n-m-1$ affine subspaces.

**Dan Ismailescu*, Geoffrey Exoo.** Hofstra University, Hempstead, NY.

*Improved lower bounds for the chromatic number of several small dimensional Euclidean spaces.*

The chromatic number of the $n$-dimensional Euclidean space, denoted $\chi(\mathbb{R}^n)$, is the minimum number of colors that can be assigned to the points of $\mathbb{R}^n$ so that no two points at distance one receive the same color. In this note, we present better lower bounds for $\chi(\mathbb{R}^n)$ for several small values of $n$.

**Jim Lawrence.** George Mason University, Fairfax, VA.

*Chains of antiprisms.*

We relate some interesting facts about sequences of polytopes, each of which is an antiprism over its predecessor.

**Carl Lee*, Clifford Taylor.** University of Kentucky, Lexington, KY.

*A generalization of the secondary polytope induced by lifting and deleting.*

Given a finite set $Q$ of points in $\mathbb{R}^d$, a regular subdivision of $\operatorname{conv} Q$ is formed by taking the convex hull of a lifting of $Q$ into $\mathbb{R}^{d+1}$ and projecting the lower hull back into $R^d$. It is well known that the poset of regular subdivisions of $Q$, ordered by refinement, is isomorphic to the face lattice of a convex polytope, the secondary polytope of $Q$, and that the coordinates of the vertices of this polytope are given by the characteristic vectors defined by Ge'lfand, Kapranov, and Zelevinsky, computed in a simple way from the volumes of the maximal simplices in the corresponding regular triangulations. We generalize some of these results by fixing an integer $0 \leq k \leq |Q|$, and for each lifting of $Q$ considering the family of subdivisions obtained by deleting in turn each of the subsets of $Q$ of size $k$. Associated with each $k$ will be a polytope with each vertex corresponding to the family of triangulations derived from a lifting. Coordinates of each vertex can be obtained by summing the characteristic vectors of the triangulations in its family. In the simple case of n points on a line and $k = 1$ we can enumerate vertices and edges of these polytopes.

**Fu Liu*, Federico Castillo.** University of California, Davis, CA.

*Ehrhart positivity for generalized permutohedra.*

The Ehrhart polynomial $i(P, m)$ of an integral polytope $P$ counts the number of lattice points in dilations of $P$. It is well known that the leading, second, and last coefficients of $i(P, m)$ are the volume of $P$, one half of the volume of the boundary of $P$ and 1, respectively, and thus are all positive. However, it is not true that all the coefficients of $i(P, m)$ are positive.

There are few families of polytopes that are known to have positive Ehrhart coefficients. De Loera *et al* conjectured that matroid polytopes have this property. In our work, we consider generalized permutohedra, which contain matroid polytopes, and conjecture they all have positive Ehrhart coefficients.

We first reduce our conjecture to another conjecture which only concerns regular permutohedra, a smaller family of polytopes. The key ingredients in the reduction are perturbation methods and a valuation on the algebra of rational pointed polyhedral cones constructed by Berline and Vergne. Then we are able to show that the third and fourth Ehrhart coefficients of regular permutohedra are always positive by explicitly computing Berline-Vergne's valuation for our polytopes. We also obtain partial results on the coefficients of the linear terms.

**Rachel E Locke, Walter Morris.*** George Mason University, Fairfax, VA.

*A Mihalisin-Klee Theorem for Fans.*

The Mihalisin-Klee Theorem states that an orientation of a 3-polytopal graph is induced by an affine function on some 3-polytope realizing the graph if the orientation is acyclic, has a unique source and a unique sink, and admits three independent monotone paths from the source to the sink. We replace the requirement that the orientation is acyclic with the assumption that it has no directed cycle contained in a face of the orientation, and show that such orientations are induced by 3-dimensional fans.

**Zokhrab Mustafaev*, Horst Martini.** University of Houston-Clear Lake, Houston, TX.

*Centered convex bodies and inequalities for cross-section measures.*

The purpose of this talk is to establish some new results on cross-section measures of centered convex bodies. More precisely, we show some connections between inequalities referring to cross-section measures and well-known affine isoperimetric inequalities. Based on this, we derive affine inequalities involving also new characterizations of ellipsoids. In addition, some related results on three-dimensional zonoids are obtained. Some of our results are also interesting from the viewpoint of the geometry of finite dimensional real Banach spaces.

**Ilya Scheidwasser.** Northeastern University, Boston, MA.

*Contractions of polygons in abstract polytopes.*

Several well-known constructions exist on abstract polytopes, such as the pyramid and prism constructions. After a brief introduction to abstract polytopes, we present two new local constructions. The first construction, called digonal contraction, allows digonal sections to be removed by merging their two edges into a single edge. The second construction, called polygonal contraction, allows polygonal sections with at least four vertices to be converted to two smaller polygons by merging two non-adjacent vertices. Neither of these contractions can be applied arbitrarily. In the case of digonal contraction, we have necessary and sufficient conditions for its use. In the case of polygonal contraction, we have necessary and sufficient conditions for its use given an assumption on the polygon, and we have necessary conditions for its use in general. We investigate when these contractions can be applied, and how polygonal contraction can be applied on a somewhat global scale in order to preserve some symmetries of the original polytope.

**Egon Schulte.** Northeastern University, Boston, MA.

*Colorful polytopes, associahedra and cyclohedra.*

Every $n$-edge colored $n$-regular graph $G$ naturally gives rise to a simple abstract $n$-polytope $P(G)$, called the colorful polytope of $G$, whose 1-skeleton is isomorphic to $G$. We describe colorful polytope versions of the associahedron and cyclohedron. Like their classical counterparts, the colorful associahedron and cyclohedron encode triangulations and flips, but now with the added feature that the diagonals of the triangulations are colored and adjacency of triangulations requires color preserving flips. The colorful associahedron and cyclohedron are derived as colorful polytopes from the edge colored graph whose vertices represent these triangulations and whose colors on edges represent the colors of flipped diagonals. Joint work with G.Araujo-Pardo, I.Hubard and D.Oliveros.

**Valeriu Soltan.** George Mason University, Fairfax, VA.

*Convex hypersurfaces with hyperplanar intersections of their homothetic copies.*

We describe all convex hypersurfaces $S$ in $\mathbb{R}^n$, possibly unbounded, such that the intersection of $S$ and any homothetic copy of $S$ lies in a hyperplane.

**Csaba D. Toth.** California State University, Northridge, CA.

*Flip distances and Hamiltonian triangulations.*

It is shown that every triangulation (maximal planar graph) on $n \geq 6$ vertices can be flipped into a Hamiltonian triangulation using a sequence of less than $n/2$ combinatorial edge flips. The previously best upper bound uses 4-connectivity as a means to establish Hamiltonicity. But in general about $3n/5$ flips are necessary to reach a 4-connected triangulation. Our result improves the upper bound on the diameter of the flip graph of combinatorial triangulations on $n$ vertices from $5.2n - 33.6$ to $5n - 33$. It is also shown that for every triangulation on $n$ vertices there is a simultaneous flip of less than $2n - 3$ edges to a 4-connected triangulation. The bound on the number of edges is tight, up to an additive constant. (Joint work with Jean Cardinal, Michael Hoffmann, Vincent Kusters, and Manuel Wettstein.)

**Gabor Tóth*, Qi Guo.** Rutgers University, Camden, NJ.

*Dual Mean Minkowski Measures and the Grünbaum Conjecture.*

For a convex body $C \subset \mathbb{R}^n$, we define two sequences $\{\sigma_{C,k}\}_{k\geq 1}$ and $\{\sigma^\circ_{C,k}\}_{k\geq 1}$ of functions on the interior of $C$. The $k$-th members are "mean Minkowski measures in dimension $k$" which are pointwise dual: $\sigma^\circ_{C,k}(O) = \sigma_{C^\circ,k}(O)$, where $O \in \operatorname{int} C$, and $C^\circ$ is the dual of $C$ with respect to $O$. We have

$$1 \leq \sigma_{C,k}(O),\ \sigma^\circ_{C,k}(O) \leq \frac{k+1}{2}.$$

The lower bound is attained if $C$ has a $k$-dimensional simplicial slice or simplicial projection. The upper bound is attained if $C$ is symmetric with respect to $O$. Klee showed that the condition $m^*_C > n - 1$ on the Minkowski measure of $C$ implies that there are $n + 1$ affinely independent affine diagonals meeting at a critical point $C^* \in C$. In 1963 Grünbaum conjectured the existence of such point in any convex body. While this conjecture remains open (and difficult), as a byproduct of the properties of the dual mean Minkowski measures, we show that

$$\frac{n}{m^*_C + 1} \leq \sigma_{C,n-1}(O^*),$$

and if sharp inequality holds then the Grünbaum conjecture holds. Our assumption is much weaker than Klee's.

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