Convex and Discrete Geometry

AMS Special Session

March 17–18, 2012, Washington, DC

Organizers: Jim Lawrence and Valeriu Soltan, George Mason University, Virginia

PROGRAM OF THE SESSION

Saturday, March 17

8:00 am	Z. Mustafaev. *One the unit ball and isoperimetrix in Minkowski spaces.*
8:30 am	J. Lawrence. *When is the product of two Davis matrices a Davis matrix?*
9:00 am	D. Oliveros. *About piercing numbers of affine planes, lines and intervals.*
9:30 am	G. Tóth. *Characterization of simplices via measures of symmetries.*
10:00am	K. Swanepoel. *Double normals in the plane and on the sphere.*
1:30 pm	H. Martini. *Minsum k-flats and minsum hyperspheres in normed spaces.*
2:00 pm	R. Dawson. *Monotone spreads of compact convex sets.*
2:30 pm	W. Kuperberg. *Small containers for large families of sets.*
3:00 pm	A. Stancu. *On the search of new affine invariants for convex bodies.*
3:30 pm	F. Fodor. *On properties of convex disc-polygons.*
4:00 pm	V. Soltan. *Convex surfaces in* $\mathbb{R}^n$ *with hyperplanar shadow-boundaries.*

Sunday, March 18

8:00 am	S. Hsiano. *Peaks and the cd-index.*
8:30 am	C. Lee. *The* cd*-indices,* CD*-vectors, and* h*-vectors of convex polytopes.*
9:00 am	G. Fejes Tóth. *Convex sets in empty convex position.*
9:30 am	E. Schulte. *Symmetric graphicahedra.*
10:00am	T. Bisztriczky. *The* $T(5)$ *property of families of overlapping unit disks.*
10:30am	S. Provan. *Minimum beam detectors for polygonal regions.*
3:00 pm	A. Bezdek. *On fair partitions of polygons.*
3:30 pm	R. Howard. *Restrictions on linear isomorphisms between spaces of smooth functions on compact manifolds.*
4:00 pm	D. Ismailescu. *A surprising geometric transformation.*
4:30 pm	J. Love. *Hom-polytopes between regular polytopes.*
5:00 pm	Seung Jin Lee. *Centrally symmetric polytopes with many faces.*

ABSTRACTS

András Bezdek. Auburn University, Auburn, AL.

On fair partitions of polygons.

A convex partition of a polygon P is a finite set of convex polygons such that the interiors of the polygons do not intersect and the union of the polygons is equal to the original polygon P. The desire to create optimal partitions of a given convex polygon furnished a number of problems in discrete geometry. The properties used in optimization among others include equal area, equal perimeter and the number of pieces. The concept of fair partitions commonly refers to problems where simultaneously several properties need to be optimized. Variations of the cake-cutting problem are the most known problems among these. This talk surveys some of the 2D and 3D results and introduces some new variants. We are particularly interested in optimization problems which are restricted to triangulations only.

András Bezdek, Wlodzimierz Holsztynski, Wlodek Kuperberg.* Auburn University, Auburn, AL.

Small containers for large families of sets.

How small can the area of a compact subset of the plane (a container) be if it contains a circle of every radius between 0 and 1? It is known that such a compact container can be nowhere dense: Holsztynski, Kuperberg, and Mycielski [MR0461433 (57#1418)] obtained a much more general result, implying that in $\mathbb{R}^n$ there exists a compact nowhere dense set that contains a translate of the boundary of every convex set of diameter $d \leq 1$. However, the question of small-area containers has not been addressed before. We present an upper bound on the minimum area of a container for circles and we discuss a number of questions for families of sets of various shapes.

Ted Bisztriczky*, K. Böröczky, A. Heppes. University of Calgary, Calgary, Canada.

The $T(5)$ property of families of overlapping unit disks.

We consider a finite family F of unit disks in the plane with the properties: $T(k)$: Any k-element subfamily of F has a (line) transversal, and $O(d)$: The distance between the centers of any two elements of F is greater than d. It is well known that F has a transversal in each of the following cases: $k = 3$ and $d = 2\sqrt{2}$ (sharp), $k = 4$ and $d = 4/\sqrt{3}$ (sharp) and $k = 5$ and $d = 2$. In this preliminary report, we present arguments that F has a transversal in the case that $k = 5$ and $d = 4/3$.

Robert Dawson. Saint Mary's University, Halifax, Canada.

Monotone spreads of compact convex sets.

By a hyperspace we shall mean a metric space, the points of which are compact convex sets in $\mathbb{R}^d$. (We may restrict our attention to, *e.g.*, convex bodies or strictly convex sets.) Such a space also has a linear structure given by scaling and the Minkowski sum. The Hausdorff "max-min" metric gives particularly interesting hyperspaces. In particular, the problem of determining which sets have the Čebyšev nearest-neighbor property is surprisingly difficult.

In a 2010 *Journal of Geometry* paper, the speaker introduced monotone arcs of bodies in hyperspaces, parametrized arcs such that, in any direction, either the support function is a monotone function of the parameter or the support points are disjoint. Such arcs have the Čebyšev property under fairly weak additional conditions. In this talk, we will look at monotone spreads, collections of convex bodies such that every two are joined within the spread by a monotone arc. We will see that these too yield examples of Čebyšev families, including some with dimension higher than any infinite-dimensional examples previously known. There is a surprising tie-in with the work of Radon, Adams, and others on linear spaces of nonsingular matrices.

Geoff Diestel, Ralph Howard*. University of South Carolina, Columbia, SC.

Restrictions on linear isomorphisms between spaces of smooth functions on compact manifolds.

Let M be a smooth compact Riemannian manifold. For a real number s the order s Sobolev norm on $C^\infty(M)$ is $\|u\|_= \left(\int_M (I+\Delta)^{s/2}u)^2 DV \right)^{1/2}$, where Δ is the Laplacian on M with positive semi-definite choice of sign and dV is the volume measure. If N is anther smooth compact Riemannian manifold, a linear map $R : C^\infty(M) \to C^\infty(N)$ is of *finite order* iff there is a constant k so that for all s there is a constant C_s with $\|Ru\|_s \leq C_s \|u\|_{s+k}$.

Theorem. If $R : C^\infty(M) \to C^\infty(N)$ is a linear isomorphism so that both R and R^{-1} have finite order, then $\dim M = \dim N$.

This result is a start on explaining why the isomorphism theorems of integral geometry are always between space of functions on manifolds of the same dimension.

Gábor Fejes Tóth. Alfréd Rényi Institute of Mathematics, Budapest, Hungary.

Convex sets in empty convex position.

We investigate the following variant of the empty n-gon problem of Erdös. Let $\mathcal{F}$ be a family of disjoint compact convex sets. A sub-family $\mathcal{F}' \subset \mathcal{F}$ is in convex position if no member $A \in \mathcal{F}'$ is contained in the convex hull of the union of the sets belonging to $\mathcal{F}' \setminus \{A\}$. $\mathcal{F}'$ is in empty convex position in $\mathcal{F}$ if it is in convex position and the convex hull of the union of its members does not contain any member of $\mathcal{F} \setminus \mathcal{F}'$. We show that for any integers $k \geq 4$ and $n \geq k$, there is an integer N such that any family of more than N disjoint compact convex sets with the property that any k members of it are in convex position has n members that are in empty convex position in the family.

Ferenc Fodor. University of Szeged, Szeged, Hungary.

On properties of convex disc-polygons.

We will consider convex disc-polygons in the plane, that is, intersections of finite families of closed unit discs. The main goal is to examine how well such disc-polygons can approximate certain classes of convex sets. We will especially concentrate on the case when the disc-polygons are generated randomly according to the uniform probability model. We will describe the asymptotic behavior of certain key geometric properties of such polygons under various smoothness conditions of the convex set being approximated. This talk contains joint results with P. Kevei (University of Szeged) and V. Vígh (University of Szeged).

Joseph Gubeladze, Jack Love*. San Francisco State University, San Francisco, CA.

Hom-polytopes between regular polytopes.

Hom-polytopes are the polytopes of affine maps between two convex polytopes. Their study is motivated by categorical analysis of polytopes–a recent trend in this classical part of geometry. First steps towards a systematic theory were recently undertaken by Bogart-Contois-Gubeladze. Currently our understanding is very limited even in the case of regular source and target polygons. We report on our joint ongoing project with J. Gubeladze on the hom-polytopes between higher dimensional regular polytopes. Counting their vertices is a blend of combinatorial, geometric, and arithmetic challenges.

Samuel Hsiao. Bard College, Annandale-on-Hudson, NY.

Peaks and the cd-index.

The chain enumerative data of a convex polytope, and more generally a regular CW-sphere, is compactly represented by its cd-index. Though the cd-index is known to be non-negative for all regular CW-spheres, a general combinatorial interpretation has been elusive. We look at conditions under which the cd-index can be interpreted as enumerating peak sets and discuss the construction of spheres for which these conditions hold.

Dan Ismailescu. Hofstra University, Hempstead, NY.

A surprising geometric transformation.

For any integer $n \geq 2$, a square can be partitioned into n^2 smaller squares via a *checkerboard-type* dissection. Does there such a "class preserving grid dissection" exist for some other types of quadrilaterals? For instance, is it true that a circumscribed quadrilateral (that is, a quadrilateral whose sides are tangent to a circle) can be partitioned into n^2 smaller circumscribed quadrilaterals, via such a "$n \times n$ grid dissection"? We prove that the answer is affirmative for every integer $n \geq 2$. Joint work with Chung-Su Hong.

Jim Lawrence. George Mason University, Fairfax, VA.

When is the product of two Davis matrices a Davis matrix?

A matrix is called a Davis matrix if the set of nonnegative elements of the linear space spanned by its columns coincides with the cone generated by its columns. The question of the title is of interest with respect to the problem of finding the sections and projections of a cone.

Carl Lee. University of Kentucky, Lexington, KY.

The **cd***-indices, CD-vectors, and h-vectors of convex polytopes.*

We will discuss connections between the **cd**-index for convex polytopes, the CD-vector of Jonathan Fine, and the toric h-vector. In particular, we will discuss formulas for converting from one to another, ways to compute them by sweeping the polytope with a hyperplane, and why the CD-vector is especially nice for simple polytopes.

Horst Martini. University of Technology in Chemnitz, Germany.

Minsum k-flats and minsum hyperspheres in normed spaces.

Given a finite set of m points (with positive weights) in an n-dimensional normed space, find k-dimensional flats having minimal sum of (weighted) distances to these given points. This class of problems contains the famous Fermat-Torricelli problem ($k = 0$) and the minsum hyperplane problem ($k = n - 1$). As a variant of the latter, we also consider minsum hyperspheres being optimal with respect to the given point set. In this talk, old and new results on these problems will be presented.

Zokhrab Mustafaev*, Horst Martini. University of Houston-Clear Lake, Houston, TX.

One the unit ball and isoperimetrix in Minkowski spaces.

One of the challenging open problems of Minkowski Geometry is that whether the unit ball must be an ellipsoid if it is a solution of the isoperimetric problem for higher-dimensional Minkowski spaces. For $d = 2$, apart from ellipses, the Radon curves have that property as well. In this talk, we discuss this problem for the Holmes-Thompson and Busemann measures in higher-dimensional Minkowski spaces.

Deborah Oliveros. Universidad Nacional Autonoma de Mexico, Mexico City, Mexico.

About piercing numbers of affine planes, lines and intervals.

In this talk we will see an interesting family of r-hypergraphs with the property, that the chromatic number is bounded from above by a function of its clique number. Bounds, that allows us to find the piercing numbers of some families of affine hyperplanes.

Scott Provan*, Marcus Brazil, Doreen Thomas, Jia Weng. University of North Carolina, Chapel Hill, NC.

Minimum beam detectors for polygonal regions.

We study the problem of finding, for any polygonally-bounded convex set S in the plane, the minimum length of a set F that is guaranteed to intersect any straight line passing through S. This has application to situations involving searching for straight line objects or blocking straight-line passage across a region. As simple as it sounds, this problem is almost completely unsolved and remarkably difficult. We provide a framework and some fundamental results for understanding the properties of minimum beam detectors–including the close connection of this problem to the Steiner tree problem–as well as some surprising examples of proposed minimum beam detectors.

Egon Schulte. Northeastern University, Boston, MA.

Symmetric graphicahedra.

Given a connected graph G with p vertices and q edges, the G-graphicahedron is a vertex-transitive simple abstract polytope of rank q whose edge graph is isomorphic to a Cayley graph of the symmetric group S_p associated with G. The G-graphicahedron is a generalization of the well-known permutahedron obtained when G is a path. Graphicahedra inherit their combinatorial symmetries from those of the underlying graphs. When G is a q-cycle, the G-graphicahedron is intimately related to the geometry of the infinite Euclidean Coxeter group $\tilde{A}_{q-1}$ and can be viewed as an edge-transitive tessellation of the $(q-1)$-torus by $(q-1)$-dimensional permutahedra, obtained as a quotient, modulo the root lattice A_{q-1}, of the Voronoi tiling for the dual root lattice A^*_{q-1} in euclidean $(q-1)$-space. This is joint work with M. Rio-Francos, I. Hubard, and D. Oliveros.

Seung Jin Lee*, Alexander Barvinok, Isabella Novik. University of Michigan, Ann Arbor, MI.

Centrally symmetric polytopes with many faces.

We consider the convex hull $\mathcal{B}_k$ of the symmetric moment curve $U(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t)$ in $\mathbb{R}^{2k}$, where t ranges over the unit circle $\mathbb{S} = \mathbb{R}/2\mathbb{Z}$. The curve $U(t)$ is locally neighborly: as long as $t_1, \ldots, t_k$ lie in an open arc of $\mathbb{S}$ of a certain length $\phi_k > 0$, the convex hull of the points $U(t_1), \ldots, U(t_k)$ is a face of $\mathcal{B}_k$. We characterize the maximum possible length ϕ_k, proving, in particular, that $\phi_k > \pi/2$ for all k. This allows us to construct centrally symmetric polytopes with a record number of faces. In this talk, I will present how to construct a d-dimensional centrally symmetric polytope P with about $3^{d/4}$ vertices such that every pair of non-antipodal vertices of P spans an edge of P.

Valeriu Soltan. George Mason University, Fairfax, VA.

Convex surfaces in $\mathbb{R}^n$ with hyperplanar shadow-boundaries.

We describe the convex surfaces in $\mathbb{R}^n$, possibly unbounded, whose shadow-boundaries with respect to parallel illumination satisfy certain hyperplanarity conditions.

Alina Stancu. Concordia University, Montreal, Canada.

On the search of new affine invariants for convex bodies.

We will present a method of producing equi-affine invariant quantities for smooth convex bodies in $\mathbb{R}^n$ and some of its applications. We will show a direction in which this method is extended to other classes of convex bodies with the purpose of obtaining new $SL(n)$-invariants and new isoperimetric type inequalities relating them.

Konrad Swanepoel. London School of Economics, London, England.

Double normals in the plane and on the sphere.

Martini and Soltan (2005) introduced the notion of a double normal of a finite set S of points: a pair $p, q \in S$ such that each point of S lies between or on the hyperplanes perpendicular to pq that pass through p and q. This notion lies between the well studied concepts of diameter pairs and antipodal pairs of a finite set of points. We investigate the problem of determining the maximum number of double normal pairs in a set of n points in Euclidean space. We show that a set of $n \geq 3$ points in the plane has at most $3\lfloor n/2 \rfloor$ double normals, which is sharp for each n. We find examples of n points in 3-space with $\frac{17}{4} - O(\sqrt{n})$ double normals. These examples all lie on a 2-sphere. Finally, we show that any set of n points on a 2-sphere has at most $\frac{17}{4} - n$ double normals, a bound that is sharp for infinitely many values of n.

Gabor A. Tóth. Rutgers University, Camden, NJ.

Characterization of simplices via measures of symmetries.

Measuring how far an n-dimensional convex body L is from an n-simplex, the interior of the convex body naturally splits into a regular and singular set. A detailed study of examples leads to the conjecture that the simplex is the only convex body with no singular points. We prove this conjecture in two specific situations: (1) L has at least n isolated extreme points; (2) There is a at point on the boundary of L.

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