Convex and Discrete Geometry

AMS Special Session

January 5-9, 2009, Washington, DC

Organizers: Wlodek Kuperberg, Auburn University, Auburn, AL,
Valeriu Soltan, George Mason University, Fairfax, VA.

PROGRAM OF THE SESSION

Monday, January 5

2:15 pm	M. Ludwig. *General affine surface areas.*
2:45 pm	J. Lawrence. *Convex polytopes with abelian vertex-transitive symmetry.*
3:15 pm	W. Weil. *Generalized averages of section and projection functions.*
3:45 pm	Z. Mustafaev. *On orthogonal chords in Minkowski spaces.*
4:15 pm	H. Martini. *Special convex sets in normed linear spaces.*
4:45 pm	K. Swanepoel. *Outer linear measure of connected sets.*
5:15 pm	W. Morris. *Finite sets as complements of finite unions of convex sets.*
5:45 pm	D. Ismailescu. *Class preserving dissections of convex polygons.*

Tuesday, January 6

1:00 pm	D. Larman. *Blocking numbers for l_p balls in three dimensions.*
1:30 pm	Z. Langi. *On the Hadwiger numbers of topological disks.*
2:00 pm	W. Kuperberg. *Unavoidable crossings.*
2:30 pm	J. Bigley Duhnham. *Extremal coin graphs in the Euclidean plane.*
3:00 pm	G. Fejes Tóth. *Shortest path among circles.*
3:30 pm	M. Naszodi. *Covering a convex set with its smaller copies.*
4:00 pm	A. Bezdek. *On a new proof of the Malfatti's problem.*

ABSTRACTS

Geir Agnarsson, Jill Bigley Dunham.* George Mason University, Fairfax, VA.

Extremal coin graphs in the Euclidean plane.

A coin graph is a simple geometric intersection graph where the vertices are represented by non-overlapping closed disks in the Euclidean plane and where two vertices are connected if their corresponding disks touch. The problem of determining the maximum number of edges of a unit coin graph on n vertices, where all the radii are of unit length, is well known and has a beautiful solution. In this talk we consider related extremal problems of coin graphs that satisfy certain natural conditions relating to the ratios of the possible radii of the coins of the graph. Further, we will explore the algebraic equations describing wheel graphs, as they relate to the maximum number of edges in our mentioned coin graphs.

Javier Alonso, Horst Martini, Zokhrab Mustafaev.* University of Houston-Clear Lake, Houston, TX.

On orthogonal chords in Minkowski spaces.

It is known that a convex plate of diameter 1 in the Euclidean plane is of constant width 1 if and only if any two perpendicular intersecting chords have total length at least 1. We show that, in general, this result cannot be extended to normed (or Minkowski) planes when the type of orthogonality is defined in the sense of Birkhoff. Inspired by this, we present also further results on intersecting chords in normed planes that are orthogonal in the sense of Birkhoff and in the sense of James.

András Bezdek*, Jan P Boronski, Wesley Brown, Braxton Carrigan, Matt Noble. Auburn University, Auburn, AL.

On a new proof of the Malfatti's problem.

The following problem was posed by Malfatti in 1803: How to arrange in a given triangle three non-overlapping circles of greatest total area? Malfatti assumed that the solution would be obtained by three mutually touching circles each touching also two edges of the triangle (commonly called as Malfatti's circles). Curiously, Malfatti had been wrong in his initial assumption. In 1930 Lob and Richmond observed that in an equilateral triangle the packing with one large inscribed triangle and two other inscribed in the remaining space is in fact better. In 1967 Goldberg outlined an argument, with graphical support, that Malfatti's arrangement never solves the area maximizing problem. It was no sooner than in 1992, when Zalgaller and Los showed that greedy arrangement is always the best (i.e. where one chooses the circles in three steps, each time choosing a maximal possible one). In the present talk, by a simple non-analytic argument, we show that the solution to the original problem must be either the Malfatti arrangement or the greedy arrangement.

Our approach can be used for the analogous question concerning perimeter, but more importantly it can be used to solve the analogous question for spherical triangles.

András Bezdek, Wlodek Kuperberg.* Auburn University, Auburn, AL.

Unavoidable crossings.

Two convex disks K and L in the plane are said to cross each other if the removal of their intersection causes each disk to fall into disjoint components. Almost all major theorems concerning the covering density of a convex disk were proved only for crossing-free coverings. A recently constructed example will be presented here, showing that, in general, all such attempts must fail. Three perpendiculars drawn from the center of a regular hexagon to its three nonadjacent sides partition the hexagon into three congruent pentagons. Obviously, the plane can be tiled by such pentagons. But a slight modification produces a (non-tiling) pentagon with an unexpected covering property: every thinnest covering of the plane by congruent copies of the modified pentagon must contain crossing pairs. The example has no bearing on the validity of Laslo Fejes Tóths bound in general, but it shows that any prospective proof must take into consideration the existence of unavoidable crossings.

Gábor Fejes Tóth. Auburn University, Auburn, AL.

Shortest path among circles.

Given a packing of open unit circles, any two points lying outside the circles at distance d from one another can be connected by a path evading the circles and having length at most $2\pi/\sqrt{27}(d-2)+\pi$. This bound cannot be improved for values of the form $2(k\sqrt{3}+1)$. Can a packing of incongruent circles with radii at most 1 force us to a greater detour? The answer is "yes", but concerning this problem we have to be satisfied with weaker upper and lower bounds for the length of the shortest path.

Paul Goodey, Wolfgang Weil.* University of Karlsruhe, Karlsruhe, Germany.

Generalized averages of section and projection functions.

A centrally symmetric star body $K \subset \mathbb{R}^d$ is known to be determined by the content of its k-dimensional central sections, $1 \leq k \leq d-1$. Groemer (1998) proved a corresponding result for arbitrary star bodies, by considering the content of half-sections. The latter gives rise to a function $s_k(K; L, v)$ on pairs (L, v), where L is a k-space and v is a unit vector in L. In 2006, we considered the average $\bar{s}_k(K; v)$ of $s_k(K; L, v)$ over all L that contain v and investigated whether the function $\bar{s}_k(K; \cdot)$ already determines K. Surprisingly, this is the case for small and large values of k, but not in general (e.g. not if $2d - 3k + 1 = 0$).

As an intermediate construction, for a fixed j-space M with $1 \leq j \leq k \leq d-1$ and $v \in M$, one may average $s_k(K; L, v)$ over all L containing M. The resulting function $\bar{s}_{jk}(K; \cdot)$ is defined on the flag manifold of pairs (M, v). Obviously, $\bar{s}_{kk}(K; \cdot) = \bar{s}_k(K; \cdot)$ and $\bar{s}_{1k}(K; \cdot) = \bar{s}_k(K; \cdot)$. We show that $\bar{s}_{jk}(K; \cdot)$ determines K uniquely, for all $2 \leq j \leq k$.

Similar results for projection functions of convex bodies will also be discussed.

Dan Ismailescu. Hofstra University, Hempstead, NY

Class preserving dissections of convex polygons.

Given a convex quadrilateral Q with a certain property $\mathcal{P}$, we are interested in finding a dissection of Q into a finite number of smaller convex quadrilaterals, each of which has property $\mathcal{P}$ as well. In particular, we prove that every cyclic (orthodiagonal, circumscribed) quadrilateral can be dissected into cyclic (orthodiagonal, respectively circumscribed) quadrilaterals. The problem becomes much more interesting if we restrict ourselves to a particular type of partition we call *grid dissection*. (Joint work with Adam Vojdany.)

Zsolt Langi. Budapest University of Technology, Budapest, Hungary.

On the Hadwiger numbers of topological disks.

The Hadwiger number $H(S)$ of a topological disk S in the plane is the maximum number of pairwise nonoverlapping translates of S that touch S. It is well known that if S is convex, then $H(S) \leq 8$. A. Bezdek, K. and W. Kuperberg conjectured that the same upper bound holds for the Hadwiger numbers of starlike disks. A. Bezdek showed that $H(S) \leq 75$ for any starlike disk S. Another question of A. Bezdek and J. Pach was whether there is a universal upper bound for the Hadwiger numbers of topological disks in general. A recent result of Cheong and Lee shows that the answer for this question is "no." In this talk, I present recent results about the Hadwiger numbers of topological disks and, in particular, about those of starlike disks.

David G. Larman. University College London, London, England.

Blocking numbers for l_p balls in three dimensions.

(Joint work with Selvinaz Szegin) The blocking number of a convex body C in Euclidean Space is the minimum number of non-overlapping translates of C which touch C and prevent, without overlapping, any other translate from touching C. A well known unsolved conjecture is that the blocking number of every convex body in 3 dimensions is at least 6. Here we show that, for l_p balls in 3 dimensions, $p < \infty$, the blocking number is at most 6.

Jim Lawrence. George Mason University, Fairfax, VA.

Convex polytopes with abelian vertex-transitive symmetry.

In studying families of mathematical objects, it is often useful to single out those having a lot of symmetry in the hope that, for these, the analysis will be made less difficult by use of the symmetry. For convex polytopes, such a simplifying assumption might be that the polytope has a vertex-transitive group of symmetries. It is curious that such polytopes having more complicated groups of symmetries are often easier to study than those having, say, abelian groups of symmetries. We consider the problem of classifying the convex polytopes P for which there is an abelian group of linear symmetries acting transitively on the set of vertices. We consider in particular the case of 4-dimensional polytopes, an investigation fruitfully begun by Z. Smilansky. Here, there is an interesting connection between the combinatorial type of the polytope and continued fractions. This is joint work with T. Bisztriczky.

Jim Lawrence, Walter Morris.* George Mason University, Fairfax, VA.

Finite sets as complements of finite unions of convex sets.

Given a finite $S \subset \mathbb{R}^d$, how many convex sets are required to write the complement as a union? Crude estimates of the number of convex sets required are given. When the restriction of openness is added, tighter bounds are obtained as an application of a theorem of Björner and Kalai. Certain families of graphs and hypergraphs connected with the problem are introduced.

Monika Ludwig. Polytechnic University, New York, NY.

General affine surface areas.

In a joint work with Matthias Reitzner (Ann. of Math., to appear), we obtained the following classification of valuations on the space, $\mathcal{K}_0^n$, of convex bodies that contain the origin in their interiors.

THEOREM. A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $SL(n)$ invariant valuation that vanishes on polytopes if and only if there is a concave function $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \phi(t)/t = 0$ such that

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K, x))\, d\mu_K(x) \tag{1}$$

for every $K \in \mathcal{K}_0^n$. Here $d\mu_K(x) = x \cdot u(K, x)\, dx$ is the cone measure on ∂K, $u(K, x)$ is the exterior normal unit vector to K at $x \in \partial K$, and

$$\kappa_0(K, x) = \frac{\kappa(K, x)}{(x \cdot u(K, x))^{n+1}},$$

where $\kappa(Kx)$ is the Gaussian curvature.

In this talk, two new families of general affine surface areas are defined. Basic properties and affine isoperimetric inequalities for these new affine surface areas as well as for the L_ϕ affine surface areas defined in (1) are discussed.

Horst Martini. Chemnitz University of Technology, Chemnitz, Germany.

Special convex sets in normed linear spaces.

It is natural to extend problems and results from classical convexity to finite dimensional normed linear spaces (Minkowski spaces). In this talk several new results in this direction will be presented, all of them related to the study of special classes of convex bodies. These results refer to new characterizations of centrally symmetric convex bodies (by using suitably defined surface area measures in Minkowski spaces), the notion of reducedness in Minkowski spaces, and convex sets in normed planes having the circular hull property (which is closely related to the concept of constant width). These new results were obtained jointly with E. Makai, G. Averkov, and M. Spirova.

Marton Naszodi University of Alberta, Edmonton, Alberta.

Covering a convex set with its smaller copies.

We consider two topics closely related to the Gohberg-Markus-Boltyanski-Hadwiger Problem, which is to prove that every convex body in $\mathbb{R}^n$ is illuminated by 2^n directions. First, we present a new equivalent formulation of the problem, and introduce a fractional version of the illumination number. We show that for symmetric convex bodies, this number is at most 2^n. As a corollary, we obtain that for any symmetric convex polytope with k vertices, there is a direction that illuminates at least $k/2^n$ vertices.

Next, we answer the following question that was posed as Problem 6 in Section 3.2 of [1]: Let H_n denote the smallest integer k such that for every convex body K in $\mathbb{R}^n$ there is a $0 < \lambda < 1$ such that K is covered by k translates of λK. Can λ be chosen independently of K; that is, is there a $0 < \lambda_n < 1$ depending on n only with the property that every convex body K in $\mathbb{R}^n$ is covered by H_n translates of $\lambda_n K$? We prove the affirmative answer.

References

[1] P. Brass, W. Moser, J. Pach, *Research problems in discrete geometry*, Springer, New York, 2005.

Konrad Swanepoel. Chemnitz University of Technology, Chemnitz, Germany.

Outer linear measure of connected sets.

We give a definition of the length of a connected set in a metric space in terms of Steiner trees on its finite subsets. We show that this length coincides with the outer linear measure of Carathéeodory (also known as 1-dimensional Hausdorff measure) restricted to connected sets. This approach yields simple proofs of theorems of Gołąb, Bógnar and Fremlin, and answers an old question of Menger on the definition of arc length. The proofs employ, apart from a modicum of graph theory, only elementary properties of connectedness, and no measure theory apart from the definition of outer linear measure.

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