# Recent Trends in Convex and Discrete Geometry

**AMS Special Session**

**January 13-14, 2006, San Antonio, Texas**

**Organizers:** Ted Bisztriczky, University of Calgary
Paul Goodey, University of Oklahoma
Valeriu Soltan, George Mason University

## PROGRAM OF THE SESSION

### Friday, January 13

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|---|---|
| 8:00 am | C. Schuett, E. Werner.* *Approximation of the Euclidean ball by polytopes.* |
| 8:30 am | M. Ghomi,* J. Choe, M. Ritore. *Relative isoperimetric inequality outside convex bodies.* |
| 9:00 am | A. Stancu. *A characterization of ellipsoids via illumination bodies.* |
| 9:30 am | M. W. Meckes. *Central limit properties of convex bodies.* |
| 10:00am | Problem session. |
| 10:30am | L. J. Schmidt,* C. Lee. *$f$-Vectors of regular triangulations.* |
| 11:00am | K. Bezdek, Z. Lángi, M. Naszódi, P. Papez.* *Ball-polytopes.* |
| 11:30am | I. Bárány, A. Pór.* *On a conjecture of Ehrhart.* |
| 1:00 pm | W. Kuperberg. *Optimal configurations of $k$ congruent balls packed in a sphere in $\mathbb{R}^n$ ($k \leq 2n$).* |
| 1:30 pm | L. Montejano. *The colorful Hadwiger transversal theorem.* (Cancelled and replaced by a problem session.) |
| 2:00 pm | A. Schuermann. *Symmetric Delone subdivisions and their application.* |
| 2:30 pm | A. Bezdek,* G. Ambrus. *Revisiting a problem of D. Ismailescu and R. Radoicic concerning dense point sets.* |
| 3:00 pm | J. Solymosi. *Additive discrete geometry.* |
| 3:30 pm | W. J. Whiteley. *Locating points in a sensor network, with distance information.* |

### Saturday, January 14

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|---|---|
| 8:00 am | P. Goodey, W, Weil.* *Determination of convex bodies by directed projection functions.* |
| 8:30 am | R. Howard,* D. Hug. *Convex bodies with constant projection functions.* |
| 9:00 am | A. Koldobsky. *On the road between intersection bodies and polar projection bodies.* |
| 9:30 am | V. Yaskin,* M. Yaskina. *Centroid bodies and comparison of volumes.* |
| 10:00am | M. Yaskina. *Non-intersection bodies all of whose central sections are intersection bodies.* |
| 10:30am | A. Zvavitch. *General measures of a convex body.* |

## ABSTRACTS

**Imre Bárány, Attila Pór,*** Case Western Reserve University, Cleveland, OH.

*On a conjecture of Ehrhart.*

Let $K$ be a convex body in the plane whose center of gravity lies at the origin. E. Ehrhart conjectured that if Area$K \geq 6$ then $K$ contains two antipodal (nonzero) lattice points. Let $M = K \cap -K$ and $g(K) = \frac{\text{Area}M}{\text{Area}K}$. Ehrhart's conjecture translates via Minkowski's theorem to $g(K) \geq \frac{2}{3}$. This was proved by Kozinec and Stewart. In this paper we give a new proof to the theorem and show the stability of $g$.

**Andras Bezdek,* Gergely Ambrus,** Auburn University, Auburn, AL.

*Revisiting a problem of D. Ismailescu and R. Radoicic concerning dense point sets.*

There are several results in the literature where one starts with a few points, describes a geometric construction to introduce some new points and proves that applying over and over the same construction one generates an everywhere dense point set of the plane. Recently D. Ismailescu and R. Radoicic (and earlier B. Grünbaum) showed that starting with a non collinear point set the repeated use of the construction "add to the figure all the intersection points of lines which connect pairs of already existing points" leads to a dense point set of the plane (with the exception of a few particular starting configurations). They also suggested to study similar problems where one uses the construction "add the incenters (circumcenters resp.) of all triangles formed by the existing points". In 2005 together with M. Iorio and M. Silva they settled these problem. Together with G.Ambrus we also solved these problems. We considered the higher dimensional versions and also proved that in cases like the circumcenter problem much more is true, namely it is enough to assume that one adds a point "close" to the circumcenter of existing triangles.

**Károly Bezdek, Zsolt Lángi, Márton Naszódi, Peter Papez,*** University of Calgary, Calgary, Canada.

*Ball-polytopes.*

The study of polytopes is one of the oldest and most well researched areas in all of mathematics. One way of looking at polytopes is to interpret them as the region bounded by intersecting hyperplanes. These hyperplanes are just surfaces of zero curvature. Suppose that we use surfaces of non-zero curvature, say of curvature one. What do we obtain by doing this? With some care we obtain ball-polytopes. Intuitively, we can think of these as fattened polytopes, but the concept is more delicate than may first appear. The aim of this talk is to survey the results obtained by our research group in the study of ball-polytopes. These results range over many different areas of geometric interest. Most results pass to higher dimensions, but we will focus on the two- and three-dimensional cases to provide insight regarding the techniques used in this field of study.

**Mohammad Ghomi,* Jaigyoung Choe, Manuel Ritore,** Georgia Institute of Technology, Atlanta, GA.

*Relative isoperimetric inequality outside convex bodies.*

We prove that the area of a hypersurface which traps a given volume outside of a convex body in Euclidean $n$-space must be greater than or equal to the area of a hemisphere trapping the given volume on one side of a hyperplane.

**Paul Goodey, Wolfgang Weil,*** Universität Karlsruhe, Karlsruhe, Germany.

*Determination of convex bodies by directed projection functions.*

We use a tensor-type integral formula for intrinsic volumes of convex bodies $K$ in $d$-dimensional Euclidean space to define a further variant of directed projection functions and show that these determine the body $K$ uniquely. We then study averages of directed projection functions and discuss the connections between the resulting integral operators and previously considered spherical transforms.

**Ralph Howard,* Daniel Hug,** University of South Carolina, Columbia, SC.

*Convex bodies with constant projection functions.*

Let $G_k(\mathbb{R}^n)$ be the Grassmannian of all $k$-dimensional subspaces of $\mathbb{R}^n$. If $K$ is a convex body in $\mathbb{R}^n$, then the *$k$-projection function* of $K$ is the function that maps $U \in G_k(\mathbb{R}^n)$ to the $k$ dimensional volume of the orthogonal projection, $K|U$, of $K$ onto $U$. When this function is constant $K$ is said to have *constant $k$-brightness.* Constant 1-brightness is the familiar case of constant width.

THEOREM. *If $n \geq 5$ and the convex body $K$ in $\mathbb{R}^n$ has constant width and constant* 3*-brightness, then $K$ is a Euclidean ball.*

The main point is that no regularity assumptions are being made about $K$.

**Alexander Koldobsky,** University of Missouri, Columbia, MO.

*On the road between intersection bodies and polar projection bodies.*

Suppose that we start with the Euclidean ball and are allowed to construct new bodies using three operations: linear transformations, p-addition and closure in the radial metric. What convex bodies can we get by this procedure? It appears that for $p = -1$ we get all intersection bodies (Goodey-Weil), and for $p = 1$ all polar projection bodies. We study the geometric structure of intermediate classes of bodies ($-1 < p < 1$).

**Wlodzimierz Kuperberg,** Auburn University, Auburn, AL.

*Optimal configurations of $k$ congruent balls packed in a sphere in $\mathbb{R}^n$ ($k \leq 2n$).*

The minimum radius of a spherical container in $\mathbb{R}^n$ that can hold $k$ unit balls ($k \leq 2n$) has been found by R. A. Rankin in 1955. For $k \leq n+1$ the configuration of the balls is unique, their centers forming the set of vertices of a $(k-1)$-dimensional regular simplex. For $k = 2n$, the configuration is unique as well, the balls' centers forming the set of vertices of a regular $n$-dimensional crosspolytope. But uniqueness does not hold in any of the remaining cases, $k = n+2, n+3, \ldots, 2n-1$. Here we present an alternate proof of Rankin's result for $n+2 \leq k \leq 2n$ that strengthens it by including a description of all of the non-unique optimal configurations, some of which exhibit traces of regularity. Also, we prove that the configuration space $\mathcal{C}_n(k)$ is connected, for every $k \leq 2n$.

**Mark W. Meckes,** Stanford University, Stanford, CA.

*Central limit properties of convex bodies.*

A number of recent papers have shown that in certain respects projections of the uniform measure on a high-dimensional convex body behave like projections of product measures. We will discuss recent Berry-Esseen-type results along these lines. The results are obtained via a novel method in convex geometry, namely Stein's method for proving probabilistic limit theorems.

**Luis Montejano,** University of Guerrero, Acapulco, Mexico.

*The colorful Hadwiger transversal theorem.*

There are multiplied or colorful versions of Helly and Caratheodory theorems in the sense of Bárány. The aim of this talk is to discuss the corresponding colorful version of the following Hadwiger transversal theorem: *Let $F = \{A_1, ..., A_n\}$ be a ordered collection of convex sets in the plane. Assume $F$ is the union of $C_1, C_2$ and $C_3$ and for every choice $A_i \in C_1, A_j \in C_2$ and $A_k \in C_3$ there is a line transversal to $A_i, A_j$ and $A_k$ consistent with the order. Then, for some $p \in \{1, 2, 3\}$ there is a line transversal to all convex sets of $C_p$.*

Hadwiger's theorem can be generalized, in the sense of Goodman and Pollack, to higher dimensions. So, we shall discuss also the colorful version of the Goodman-Pollack hyperplane transversal theorem.

**Laura J. Schmidt,* Carl Lee,** University of Wisconsin-Stout, Menomonie, WI.

*f-Vectors of regular triangulations.*

Billera and Lee describe a set of necessary conditions for $f$-vectors of antistars in simplicial polytopes, and hence for regular triangulations and (by duality) for unbounded, simple polyhedra. It is not yet known whether these conditions are sufficient. In joint work with Carl Lee we construct certain classes of regular triangulations to demonstrate the sufficiency of these conditions in low dimensions. The construction exploits some of the combinatorial structure of the simplicial polytopes used in the proof of the $g$-Theorem.

**Achill Schuermann,** University of Magdeburg, Magdeburg, Germany.

*Symmetric Delone subdivisions and their application.*

A classical topic in Discrete Geometry with various applications to other fields are Delone subdivisions of discrete point sets. In two classical papers, Voronoi developed a theory of $L$-types, which classifies geometric lattices in $d$-dimensional Euclidean spaces according to their Delone subdivision or Dirichlet-Voronoi cell respectively. For a given dimension $d$ such a classification can theoretically be used to solve the lattice covering problem. This is impratical though for $d \geq 6$. We therefore describe an extension of the classical theory, which enables us to classify Delone subdivisions with certain prescribed properties. Using convex optimization techniques, we apply the new theory to compute various new best known covering lattices. Moreover, we show how to use the theory to obtain a complete classification of totally real thin number fields.

**Carsten Schuett, Elisabeth Werner,*** Case Western Reserve University, Cleveland, OH.

*Approximation of the Euclidean ball by polytopes.*

We prove a result on approximation of the Euclidean ball by polytopes with a fixed number of vertices.

**Jozsef Solymosi,** University of British Columbia, Vancouver, Canada.

*Additive discrete geometry.*

In this talk we show examples how to apply results from additive number theory to discrete geometry.

**Alina Stancu,** University of Massachusetts, Lowell, MA.

*A characterization of ellipsoids via illumination bodies.*

Let $K \subset \mathbb{R}^n$ be a convex body and let $\delta > 0$ be a real number. We call the $\delta$-illumination body associated to $K$ the convex body $K^\delta = \{x \in \mathbb{R}^n : Vol_n(co[x, K] \setminus K) \leq \delta\}$. The subject of this talk is the conjecture stating that $K$ is homothetic to $K^\delta$ if and only if $K$ is an ellipsoid.

**Walter J. Whiteley,** York University, Toronto, Canada.

*Locating points in a sensor network, with distance information.*

In many applications, there is a basic geometric and combinatorial problem: do we have enough geometric data (pairwise distances) to locate all of the objects uniquely? Recent results of Jackson and Jordan characterize which graphs of distances $G$ in the plane give unique locations, for 'generic' configurations. We present some new results on graphs $G$ in the plane such that the 'square graph' $G^2$ (adding edges between neighbors of each vertex in $G$) gives this generic global rigidity. These are graphs which are (i) connected and (ii) removing a single edge can only separate a single vertex, not two larger components. The extension to 3-space is: $G^3$ gives generic global rigidity if and only if $G$ is (i) connected, and (ii) removing any 2-valent vertex can only separate a single vertex, not two larger components.

We close with a set of new conjectures about globally rigid graphs in adjacent dimensions ($n$-space and $n + 1$-space), using the techniques of coning, and raising the power. The initial results evolved from joint work with Brian Anderson (Australian National University), David Goldenberg, Stephen Morse, Richard Yang (Yale), Tolga Eren and Peter

Belhumeur (Columbia). The new results include joint work with Matthew Cheung, York University.

**Vladyslav Yaskin,* Maryna Yaskina,** University of Missouri, Columbia, MO.

*Centroid bodies and comparison of volumes.*

For $-1 < p < 1$ we introduce the concept of a polar $p$-centroid body $\Gamma_p^* K$ of a star body $K$. We consider the question of whether $\Gamma_p^* K \subset \Gamma_p^* L$ implies $\mathrm{vol}(L) \leq \mathrm{vol}(K)$. Our results extend the studies by Lutwak in the case $p = 1$ and Grinberg, Zhang in the case $p > 1$.

**Maryna Yaskina,** University of Missouri, Columbia, MO.

*Non-intersection bodies all of whose central sections are intersection bodies.*

We construct symmetric convex bodies that are not intersection bodies, but all of their central hyperplane sections are intersection bodies. This result extends the studies by Weil in the case of zonoids and by Neyman in the case of subspaces of $L_p$.

**Artem Zvavitch,** Kent State University, Kent, OH.

*General measures of a convex body.*

In this talk we will show how different results and formulas in Convex Geometry known only for the case of Volume measure can be adapted to the case of a general measure. We will also consider a number of inequalities for general measures which turned to be useful to prove new results for a regular volume case.

## 1. Shades and convexity of hypersurfaces

Let $M \subset \mathbb{R}^{n+1}$ be a smooth hypersurface homeomorphic to the sphere $\mathbf{S}^n$, and $n\colon M \to \mathbf{S}^n$ be a unit normal vector field, or the Gauss map of $M$. For every unit vector $u \in \mathbf{S}^n$, the corresponding *shade* cast on $M$ is defined by

$$S_u := \{\, p \in M \mid \langle n(p), u \rangle > 0 \,\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. Suppose that $S_u$ is homeomorphic to a ball for each $u$.

**Problem.** Does it then follow that $M$ is convex, i.e., it bounds a convex body? More generally, is connectedness of each shade $S_u$ enough to ensure convexity of $M$?

For $n = 1$, it is an easy exercise to show that the answer is yes. For $n = 2$, the answer is also yes as was proved in [1]. For $n \geq 3$, however, the answer is not known.

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Mohammad Ghomi, Georgia Institute of Technology

## 2. Sizes of projections of convex bodies

**Problem 1.** Let $K_1,\ K_2$ be convex bodies in $\mathbb{E}^4$. Assume that, for every $u \in \mathbf{S}^3$, the surface area of the projection $K_1|u^\perp$ is the same as that of $K_2|u^\perp$. Does it follow that, for every 2-dimensional subspace $E$ of $\mathbb{E}^4$, the area of $K_1|E$ equals that of $K_2|E$?

The result is true both for centrally symmetric bodies and for polytopes. The reverse implication is always true and is a consequence of the Cauchy-Kubota formulas. A more general formulation of the problem, using intrinsic volumes $V_i$, is the following.

**Problem 2.** Let $K_1,\ K_2$ be convex bodies in $\mathbb{E}^n$ and, for $j \in \{2, \ldots, n-1\}$, denote by $G(n, j)$ the Grassmannian of $j$-dimensional subspaces of $\mathbb{E}^n$. If for some $2 \leq i < j$ we have $V_i(K_1|E) = V_i(K_2|E)$ for all $E \in G(n, j)$, does it follow that $V_i(K_1|F) = V_i(K_2|F)$ for all $F \in G(n, i)$?

An early version of these problems can be traced back to Minkowski [4] and this was taken up by Firey [1]. An overview of the general area is presented in [3] and the details of the polytopal case can be found in [2].

References

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Paul Goodey, University of Oklahoma

## 3. The extreme points of the collection of sets of constant width

Let $\mathcal{C}^n$ be the collection of all compact convex subsets of $\mathbb{R}^n$ modulo translations. Then is two compact convex sets are identified if they are translates of each other. To make this a bit more precise, for $K \in \mathcal{C}^n$ let $h_K$ be the support function of $K$ viewed as function on the unit sphere $\mathbf{S}^{n-1}$. Let $C(\mathbf{S}^{n-1})$ be the Banach space of continuous real valued functions on $\mathbf{S}^{n-1}$. Let $\mathcal{L}$ be the subspace of $C(\mathbf{S}^{n-1})$ consisting of the restrictions of linear functions to $\mathbf{S}^{n-1}$. Then $\mathcal{L}$ is an $n$-dimensional closed subspace of $C(\mathbf{S}^{n-1})$. Two compact convex sets differ by a translation if and only if the difference of their support function is in $\mathcal{L}$. Let $C(\mathbf{S}^{n-1})/\mathcal{L}$ be the quotient space of $C(\mathbf{S}^{n-1})$ by $\mathcal{L}$. Then the map $K \mapsto h_K + \mathcal{L}$ gives a bijection of $\mathcal{C}^n$ with the set $\{h_K : K \in \mathcal{C}^n\}/\mathcal{L}$. Minkowski sums of convex sets corresponds to addition of support functions. Thus $\{h_K : K \in \mathcal{C}^n\}/\mathcal{L}$ is a convex subset of $C(\mathbf{S}^{n-1})/\mathcal{L}$.

Let $\mathcal{W}^n$ be the subset of $\mathcal{C}^n$ of compact convex subsets of constant width 2. Then it is not hard to check that $\mathcal{W}^n$ is a compact convex subset of $C(\mathbf{S}^{n-1})/\mathcal{L}$. Therefore, by the Krein-Milman Theorem, it is the closed convex hull of its extreme points.

**Problem:** *What are the extreme points of $\mathcal{W}^n$?*

In two dimensions this is not hard to answer. Identify $\mathbf{S}^1$ with the quotient $\mathbb{R}/2\pi\mathbf{Z}$. Then $C(\mathbf{S}^1)$ is identified with the $2\pi$ periodic functions on $\mathbb{R}$ and $\mathcal{L} = \operatorname{Span}\{\cos(\theta), \sin(\theta)\}$. The map $\theta \mapsto \theta + \pi$ corresponds to the antipodal map of $\mathbf{S}^1$. Let $p \colon \mathbf{S}^1 \to \{-1, 1\}$ be measurable, odd with respect to the antipodal map (that is $p(\theta+\pi) = -p(\theta)$) and satisfy

$$\int_0^{2\pi} p(\theta)\cos(\theta)\, d\theta = \int_0^{2\pi} p(\theta)\sin(\theta)\, d\theta = 0. \tag{1}$$

Then, using Fourier series or the solution to the 2-dimensional Minkowski problem, it can be seen there is a unique solution, $h$, to $h''(\theta) + h(\theta) = 1 + p(\theta)$. This $h$ will be the support function of a planar convex body that is an extreme point of $\mathcal{W}^2$, and all extreme points of $\mathcal{W}^2$ are of this form. To make this somewhat more geometric consider an odd two valued step function $p \colon \mathbf{S}^1 \to \{-1, 1\}$ that satisfies (1). Then the corresponding convex body will be a Reuleaux figure as in the figure below. The set all such Reuleaux figures is dense in the set of all extreme points of $\mathcal{W}^2$.

To the best of my knowledge there is no such description of a dense subset of the set of extreme points of $\mathcal{W}^n$ for $n \geq 3$.

One motivation for finding the extreme points of $\mathcal{W}^n$ is trying to generalize the Blaschke-Lebesgue theorem to higher dimensions. The problem is to find the element of $\mathcal{W}^n$ of least volume. When $n = 2$ the result of Blaschke and Lebesgue is that this minimizer is the Reuleaux triangle. Here is a passable approach to this question. If $V(K)$ is the volume of $K$, then the Brunn-Minkowski inequality implies that $K \mapsto V(K)^{1/n}$ is a concave function on $\mathcal{W}^n$. Therefore a minimizer of $V(\cdot)$ on $\mathcal{W}^n$ will occur at an extreme point of $\mathcal{W}^n$. And in two dimensions one way to prove the Blaschke-Lebesgue is to check that of all Reuleaux figures, the Reuleaux triangle triangle has the least area. Hopefully knowing the extreme points of $\mathcal{W}^n$ would also be useful in generalizing other extremal problems for sets of constant width.

Here are a few references related to this problem. For sets of constant width, Reuleaux figure, and related topics see the excellent survey article [3] and the references therein. For recent results related to the Blaschke-Lebesgue Theorem, attempts to generalize it to higher dimensions, and some related problems see [1, 2, 4].

References


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Ralph Howard, University of South Carolina


## 4. Monotonicity of volumes of random simplices

For a convex body $K \subset \mathbb{R}^n$, let

$$M(K) = \frac{1}{\mathrm{vol}_n(K)^{n+1}} \int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{x_1, \dots, x_{n+1}\}) dx_1 \cdots dx_{n+1}$$

denote the mean volume of a random simplex in $K$.

**Conjecture:** If $K_1, K_2 \subset \mathbb{R}^n$ are convex bodies with $K_1 \subset K_2$, then $M(K_1) \leq M(K_2)$.

**Weaker Conjecture:** If $K_1, K_2 \subset \mathbb{R}^n$ are convex bodies with $K_1 \subset K_2$, then $M(K_1) \leq a^n M(K_2)$, for some universal constant $a$.

The motivation for the conjectures comes from the slicing problem. Recall that the slicing problem asks whether the isotropic constant $L_K$ of every convex body $K$ is bounded above by some absolute constant $c$. The connection is through the formula

$$L_K \approx \left(\frac{M(K)}{\mathrm{vol}_n(K)}\right)^{1/n} \sqrt{n}$$

(see for example A. Giannopoulos [1]). In fact, combined with a recent result of B. Klartag (Theorem 1.1 of [2]), this formula and the weaker conjecture would give a positive solution to the slicing problem. Furthermore, if a still weaker version of the conjecture can be proved with $a = a(n)$, one obtains $L_K \leq ca(n)$. The best known universal bound is $L_K \leq cn^{1/4}$ (Corollary 1.2 in the same paper of Klartag).

A potential advantage of this approach to the slicing problem, aside from the geometric plausibility of the conjectures, is the affine invariance of $M(K)$, as opposed to the more usual expressions for $L_K$ which are variational or require fixing a position of $K$.

References

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Mark W. Meckes, Stanford University

## 5. Sphere covering

Roughly speaking, the sphere covering problem asks to determine most economical ways to cover $\mathbb{R}^d$ with equally sized spheres. To give a precise description, we consider a discrete set $\Lambda \subset \mathbb{R}^d$ and its *point density*

$$\mathrm{dens}(\Lambda) = \limsup_{\lambda \to \infty} \frac{\lambda^d \cdot \mathrm{vol}(B^d)}{\mathrm{card}(\Lambda \cap \lambda B^d)},$$

where $B^d$ is the unit ball of $\mathbb{R}^d$. If $\Lambda$ is a lattice (a discrete subgroup) of $\mathbb{R}^d$, then $\mathrm{dens}(\Lambda)$ is equal to the *determinant* $\det(\Lambda)$ of the lattice, i.e. the volume of a fundamental domain of $\Lambda$. The *covering radius* of $\Lambda$ is given by

$$\mu(\Lambda) = \inf\{\mu > 0 : \Lambda + \mu B^d = \mathbb{R}^d\}$$

and the *covering density* of $\Lambda$ is defined by

$$\Theta(\Lambda) = \frac{\mu(\Lambda)^d}{\mathrm{dens}(\Lambda)} \cdot \mathrm{vol} B^d.$$

Note that the covering density is invariant with respect to similarities of $\mathbb{R}^d$.

**Sphere Covering Problem:** Determine $\Theta_d = \min_{\Lambda} \Theta(\Lambda)$ and discrete sets in $\mathbb{R}^d$ attaining it.

An introduction, together with many references is given in the highly recommended book by Conway and Sloane [2]. In dimension $d = 2$ the problem has been solved by Kershner [5]. Here the optimal sphere covering is attained by the hexagonal lattice. In dimensions $d > 2$ the answer to the general sphere covering problem is unknown. Nevertheless, the problem has been solved with the restriction to lattices for dimensions $d = 3, 4, 5$ (cf. [1,2,6]). Recently we found new best known sphere coverings for dimensions $d = 6, \ldots, 21$ (see [4]). For an updated list with additional information on the involved lattices we refer to our webpage [7]. The following are challenging open sphere covering problems.

**Problem 1.** ("Kepler analogue") Prove or disprove covering optimality of the bcc-lattice ($\mathsf{A}_3^*$, cf. [2]).

**Problem 2.** Prove or disprove lattice covering optimality of the recently found lattices $\mathsf{L}_6^c$ for $d = 6$ and $\mathsf{L}_7^c$ for $d = 7$ (cf. [7] for details).

**Problem 3.** Prove lattice covering optimality of the Leech lattice for $d = 24$ (cf. [2]).

**Problem 4.** Find asymptotically good coverings. How close can the covering density come to the lower bound by Coxeter, Few and Rogers (cf. [2]). It is obtained by considering an "ideal covering" with sphere centers being the vertices of a simplicial subdivision of $\mathbb{R}^d$, consisting of regular simplices only.

**Problem 5.** Find non–lattice coverings, which are less dense than any lattice covering in their dimension.

**Problem 6.** Find new best known (lattice) sphere coverings for $d \geq 6$.

## References


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Achill Schürmann, University of Magdeburg

## 6. Sections and projections of homothetic convex sets

Let $K_1, K_2 \subset \mathbb{R}^n$ be closed convex sets, possibly unbounded.

**Problem.** Is it true that $K_1$ and $K_2$ are homothetic if and only if either of the conditions $(a), (b)$ below holds?

$(a)$ The orthogonal projections of $K_1$ and $K_2$ on each 2-dimensional plane are homothetic.

$(b)$ There are points $p_1, p_2 \in \mathbb{R}^n$ such that for every pair of parallel 2-dimensional planes $L_1$ and $L_2$ through $p_1$ and $p_2$, respectively, the sections $K_1 \cap L_1$ and $K_2 \cap L_2$ are both empty or homothetic.

This problem has a well-known affirmative solution when both $K_1$ and $K_2$ are compact (see Rogers [2] and Burton [1]). The case when $K_1$ and $K_2$ are translates of one another is considered in Soltan [3].

References

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## 7. Triple points determined by unit circles

Given a family of unit circles on the plane, a *triple point* is that incident to at least three circles from the family.

**Problem 1.** What is the maximum number of triple points determined by $n$ pairwise distinct unit circles in the plane?

**Problem 2.** Prove that the maximum number of triple points determined by $n$ pairwise distinct unit circles is $o(n^2)$.

On the other hand, I don't know how to find $n$ pairwise distinct unit circles that determine at least $n^{1+\epsilon}$ triple points.

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## 8. Strictly convex indecomposable bodies

Let $n \geq 3$. A convex body $K \subset \mathbb{R}^n$ is strictly convex, if it does not contain any segments in the boundary, and $K$ is indecomposable, if a decomposition as a Minkowski sum $K = M + L$ (with convex bodies $M, L$) implies $M = \alpha K + x$ and $L = \beta K + y$, for some $\alpha, \beta \geq 0$ and $x, y \in \mathbb{R}^n$. Both, the set of all strictly convex bodies in $\mathbb{R}^n$ and the set of indecomposable convex bodies, are dense $G_\delta$-sets in the set $\mathcal{K}^n$ of all convex bodies (e.g. polytopes with triangular 2-faces are indecomposable). Let $\mathcal{I}^n$ be the set of convex bodies $K$, which are strictly convex **and** indecomposable. As an intersection of two dense $G_\delta$-sets, $\mathcal{I}^n$ is a dense $G_\delta$-set in $\mathcal{K}^n$.

The problem is to find/describe one element of $\mathcal{I}^n$.

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## 9. Area discrepancy of unimodular triangulations

It was first shown by Monsky [3] (see also Stein & Szabó [4]) that a square cannot be dissected into an odd number $2n+1$ of equal-area triangles. Monsky's proof is algebraic, via 2-adic valuations, and does not provide an estimate for the minimum discrepancy of odd dissections,

$$f(2n+1) \ := \min_{\Delta \in \mathcal{D}_{2n+1}} |\max_{\sigma \in \Delta} A(\sigma) - \min_{\sigma \in \Delta} A(\sigma)|,$$

where $\mathcal{D}_{2n+1}$ denotes the set of all dissections of the unit square into $n+1$ triangles, and $A(\sigma)$ is the area of the triangle $\sigma$.

The problem is to determine the asymptotics of $f(2n+1)$. It is clear that $f(2n+1) \leq \frac{1}{2n} - \frac{1}{2(n+1)} = \frac{1}{2n(n+1)}$. Experiments by Mansow [2] (based on a combinatorial enumeration of triangulations, and numerical minimization of the discrepancy) suggest that $f(2n+1)$ decreases singly-exponentially:

$$\begin{aligned} f(3) &= 0.25 \\ f(5) &\leq 0.0225 \\ f(7) &\leq 0.0031 \\ f(9) &\leq 0.00014 \\ f(11) &\leq 0.00000415. \end{aligned}$$

At the same time, gap theorems from semialgebraic geometry (see Basu et al. [1, S. 13.2]) together with Monsky's result and with the finiteness of the number of combinatorial types imply a doubly exponential lower bound on $f(2n+1)$.

References


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