Convexity and Combinatorics

AMS Special Session

November 6–7, 2010, Richmond, VA

Organizers: Jim Lawrence and Valeriu Soltan, George Mason University, Virginia

PROGRAM OF THE SESSION

Saturday, November 6

8:00 am	G. Agnarsson. *On Minkowski sum of simplices and their flags.*
8:30 am	F. Liu. *Higher integrality conditions and volumes of slices.*
9:00 am	H. Martini. *Some topics and notions from convexity extended to normed linear spaces.*
9:30 am	M. Breen. *Some combinatorial results for staircase visibility.*
10:00am	C. Nicolas. *Interval decompositions of k-edges and applications.*
10:30am	R. Vitalie. *Intrinsic volumes.*
2:30 pm	W. Kuperberg. *The set of packing and covering densities of convex disks.*
3:00 pm	Z. Mustafaev. *Cross-section measures and their applications in Minkowski spaces.*
3:30 pm	D. Oliveros. *Helly-type theorems and intersection graphs: one nice relation.*
4:00 pm	E. Schulte. *Regular polyhedra of index two in space.*
4:30 pm	M. Naszódi. *On the maximal distance between convex bodies in $\mathbb{R}^n$.*
5:00 pm	V. Soltan. *Convex surfaces with planar quadric sections.*

Sunday, November 7

8:00 am	M. Omar. *Permutation polytopes and Ehrhart polynomials.*
8:30 am	R. Howard. *Tangent cones and regularity of convex sets.*
9:00 am	D. Caraballo. *Convexity and geometric measure theory.*
9:30 am	T. Bisztriczky. *The $T(5)$ property of families of overlapping unit disks.*
10:00am	G. Tóth. *A measure of symmetry for convex sets and its application to moduli for minimal immersions of spheres.*
10:30am	A. Bezdek. *Thin covering of the sphere with various convex spherical sets.*
2:30 pm	E. Morales. *Only solid spheres admit a false axis of revolution.*
3:00 pm	M. Mossinghoff. *Enumerating isodiametric and isoperimetric polygons.*
3:30 pm	W. Morris. *Oriented matroids and intersection of simplices generated by a family of segments.*
4:00 pm	C. Tóth. *Convex subdivisions for point sets.*
4:30 pm	J. Mihalisin. *Near cubes: a relatively simple family of simple relatives of the cube.* (canceled)
5:00 pm	J. Lawrence. *Intersections of descending sequences of affinely equivalent polytopes.*

ABSTRACTS

Geir Agnarsson. George Mason University, Fairfax, VA.

On Minkowski sum of simplices and their flags.

We consider a Minkowski sum of k standard simplices in $\mathbb{R}^r$ and its chains of faces, for given $k, r \in \mathbb{N}$. We define its flag polynomial in a direct and canonical way in terms of the *k-th master polytope* $P(k)$. This polynomial is related to the well-known flag vector, and it has some nice algebraic properties that one can use to obtain explicit formulae for the number of chains of faces of fixed dimensions and height.

András Bezdek. Auburn University, Auburn, AL.

Thin covering of the sphere with various convex spherical sets.

One of the basic problems in discrete geometry is to determine the most efficient packing or covering of a given convex set in the plane, in the space or on the sphere. This talk will concentrate on coverings of the surface of the unit sphere S^2 (in three dimensional space). In case of a given convex spherical set, one wants to find the smallest number of congruent copies needed to cover S^2. This talk will describe a new family of convex spherical sets, which do not tile S^2, yet for which the optimal coverings can be determined. These convex sets also have an unexpected covering property: no rearrangement of the sets taking part of the covering can produce a crossing free covering (we say that two spherical discs cross each other if the removal of their intersection causes each disk to fall into disjoint components). These results were motivated by the construction and the proof technique used in a recent joint paper with W. Kuperberg: Unavoidable crossings in the thinnest plane covering with congruent convex discs (Discrete Comput. Geom. (2010) 43: 187-208).

Ted Bisztriczky. University of Calgary, Calgary, Alberta, Canada.

The $T(5)$ property of families of overlapping unit disks.

We consider a finite family F of unit disks in the plane with the properties. $T(k)$: Any k-element subfamily of F has a (line) transversal, and $O(d)$: The distance between the centers of any two elements of F is greater than d. It is well known that F has a transversal in each of the following cases: $k = 3$ and $d = 2\sqrt{2}$ (sharp), $k = 4$ and $d = 4/\sqrt{3}$ (sharp), and $k = 5$ and $d = 2$. In this preliminary report, we present arguments that F has a transversal in the case that $k = 5$ and $d = \sqrt{3}$.

Marilyn Breen. University of Oklahoma, Norman, OK.

Some combinatorial results for staircase visibility.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that employ the idea of visibility via staircase paths. Here we present several examples of these analogues, including the following: In the plane, let C be an orthogonal polygon bounded by a simple closed curve, and assume that C is starshaped via staircase paths.

Let P be a set in the complement of the interior of C, $\operatorname{int} C$. If every 4 points of P see a boundary point of C via staircase paths in the complement of $\operatorname{int} C$, then there is a boundary point b of C such that every point of P sees b (via staircase paths in the complement of $\operatorname{int} C$). The number 4 is best possible, even if C is convex via staircase paths.

David George Caraballo. Georgetown University, Washington, DC.

Convexity and geometric measure theory.

In this talk, I will present my recent work establishing strong, new connections between geometric measure theory and results concerning convexity theory which have found wide application in fields such as functional analysis, economics, optimization, and control theory. One of the most important and well-known properties of convex sets is the fact that a closed subset K of $\mathbb{R}^n$ with non-empty interior is convex if and only if it has a supporting hyperplane through each point of its topological boundary. I have refined this result, showing that such a set K is convex if and only if it has a supporting hyperplane through each point of its reduced boundary, which may be much smaller than the topological boundary. This is surprising as it is not at all clear why the reduced boundary from geometric measure theory should contain all the convexity information about a closed subset of $\mathbb{R}^n$ with non-empty interior. I similarly refined a standard separation theorem, as well as a representation theorem for convex sets, and extended each result to other notions of boundary from the literature, deducing the corresponding classical results from convex analysis as special cases.

Jesus De Loera, Katherine Jones, Mohamed Omar.* University of California, Davis, CA.

Permutation polytopes and Ehrhart polynomials.

Permutation polytopes are convex hulls of real representations of finite groups. The Birkhoff polytope is a classical example, and it is well known that this polytope is the convex hull of doubly stochastic matrices. Of particular interest has been the study of volumes of the Birkhoff polytope and its faces. We study this in a more general context by studying Ehrhart polynomials of general permutation polytopes, providing an intimate interplay between convex geometry, group theory, and optimization.

Mohammad Ghomi, Ralph Howard.* University of South Carolina, Columbia, SC.

Tangent cones and regularity of convex sets.

Let X be a locally closed subset of $\mathbb{R}^n$ so that all the tangent cones (in the sense of Federer), T_pX, are affine hypersurfaces of $\mathbb{R}^n$, the dependence on p is continuous, and the measure theoretic multiplicity at each point is at most $m < 3/2$. Then X is an embedded C^1 hypersurface of $\mathbb{R}^n$. This is used to show: (1) any convex real analytic hypersurface of $\mathbb{R}^n$ is C^1 and (2) if X is real algebraic, strictly convex, and unbounded, then it is a graph of a C^1 function over a hyperplane.

Wlodek Kuperberg. Auburn University, Auburn, AL.

The set of packing and covering densities of convex disks.

For each convex disk K (a convex compact subset of the plane, with a non-void interior), its packing density $\delta(K)$ and covering density $\vartheta(K)$ form an ordered pair of real numbers, i.e., a point on the coordinate plane. The set Ω, consisting of points assigned this way to all convex disks, is the subject of this talk. A few known inequalities on $\delta(K)$ and $\vartheta(K)$ jointly outline a relatively small convex polygon that contains Ω, but the exact shape of Ω remains a mystery. We present this polygonal region, and then we explicitly exhibit a certain convex region contained in Ω and occupying a good portion of it.

Jim Lawrence. George Mason University, Fairfax, VA.

Intersections of descending sequences of affinely equivalent polytopes.

In 1952 Borovikov published a proof of the conjecture of Kolmogorov that the intersection of a descending sequence of simplexes in Euclidean space must be a simplex. We consider the following question, analogous to Kolmogorov's: What are the possibilities for the intersection of a descending sequence of compact convex sets, each of which is affinely equivalent to a given compact convex set? The answer to this question involves the notion of an "affine retract" and yields a generalization of the result of Borovikov.

Fu Liu. University of California, Davis, CA

Higher integrality conditions and volumes of slices.

A polytope is integral if all of its vertices are lattice points. The constant term of the Ehrhart polynomial of an integral polytope is known to be 1. I generalize this result by introducing the definition of k-integral polytopes, where 0-integral is equivalent to integral. I will show that the Ehrhart polynomial of a k-integral polytope P has the properties that the coefficients in degrees of less than or equal to k are determined by a projection of P, and the coefficients in higher degrees are determined by slices of P. A key step of the proof is that under certain generality conditions, the volume of a polytope is equal to the sum of volumes of slices of the polytope.

Horst Martini. Chemnitz University of Technology, Chemnitz, Germany.

Some topics and notions from convexity extended to normed linear spaces.

In this talk we will discuss some basic topics and notions from classical convexity, e.g. cross-section measures and special classes of convex bodies (like ellipsoids, centrally symmetric bodies or reduced bodies). We will present their role in the geometry of finite dimensional Banach spaces, also called Minkowski geometry, by giving examples how these topics and notions from convexity theory yield new results in Minkowski geometry, as tools and also as studied objects.

James E. Mihalisin. Meredith College, Raleigh, NC.

Near cubes: a relatively simple family of simple relatives of the cube.

The family of "near cubes" will be defined. Simply put—near cubes are the convex polytopes and non-convex polytopal complexes that are obtainable by performing a sequence of "clique swaps" on the d-cube. A "clique-swap" is just the dual version of a bi-stellar flip. The combinatorial symmetries of the near cubes will be determined and the non-convexity of some of them will be discussed. As time permits and audience interest allows, the connection between clique swaps and the Gale transform will be discussed and the general utility of clique swaps will be debated.

Efren Morales*, Jesus Jeronimo, Luis Montejano. Universidad Autonoma de Guerrero, Acapulco, Mexico.

Only solid spheres admit a false axis of revolution.

Let $K \subset \mathbb{R}^3$ be a convex body. A point p_0 is a point of revolution for K if every planar section of K through p_0 has an axis of symmetry that passes through p_0. In particular, every point that lies in an axis of revolution is a point of revolution. A line $L \subset \mathbb{R}^3$ is a *false axis of revolution*, if every point of L is a point of revolution for K but L is not an axis of revolution. The purpose of this paper is to prove that only solid spheres admit a false axis of revolution.

Walter Morris. George Mason University, Fairfax, VA.

Oriented matroids and intersection of simplices generated by a family of segments.

Let $\mathcal{S} = \{S_1, \ldots, S_d\}$ be a collection of segments in $\mathbb{R}^d$. $\mathcal{S}$ has property P if every set of points $\{x_1, \ldots, x_d\}$ with $x_i \in S_i$ for all $i = 1, \ldots, d$ is affinely independent. If, in addition, there is a point $p \in \mathbb{R}^d$ so that $p \in \operatorname{int}(\operatorname{conv}\{x_1, \ldots, x_d\})$ whenever $x_i \in S_i$ for all $i = 1, \ldots, d$, then $\mathcal{S}$ is said to have property K. We give some sufficient conditions for an oriented matroid analog of property K to hold.

Michael J. Mossinghoff. Davidson College, Davidson, NC.

Enumerating isodiametric and isoperimetric polygons.

For a positive integer n that is not a power of 2, precisely the same family of convex polygons with n sides is optimal in three different geometric problems. These polygons have maximal perimeter relative to their diameter, maximal width relative to their diameter, and maximal width relative to their perimeter. We study the number of different convex n-gons $E(n)$ which are extremal in these three isodiametric and isoperimetric problems. We show that $E(n) > \frac{p}{4n} 2^{n/p}$ if p is the smallest odd prime divisor of n, prove that $E(n) = 1$ if and only if $n = p$ or $n = 2p$ for some odd prime p, and compute the exact value of $E(n)$ in several cases.

Zokhrab Mustafaev*, Horst Martini. University of Houston-Clear Lake, Houston, TX.

Cross-section measures and their applications in Minkowski spaces.

We continue to investigate extremal values of inner and outer radii of the unit ball in Minkowski spaces (i.e., finite dimensional real Banach spaces) for the Holmes-Thompson and Busemann measures. Furthermore, we give a related new characterization of ellipsoids in $\mathbb{R}^d$ via codimensional cross-section measures.

Márton Naszódi. Eötvös Loránd University, Budapest, Hungary.

On the maximal distance between convex bodies in $\mathbb{R}^n$.

We consider the following version of the Banach-Mazur distance of (non-symmetric) convex bodies in $\mathbb{R}^n$ which was introduced by Grünbaum:

$$d(K,L) = \inf\{|\lambda| : \lambda \in \mathbb{R},\ \tilde{K} \subset \tilde{L} \subset \lambda\tilde{K}\},$$

where the infimum is taken over all non-degenerate affine images $\tilde{K}$ and $\tilde{L}$ of K and L. Gordon, Litvak, Meyer and Pajor showed that for any two convex bodies $d(K,L) < n$, moreover, if K is a simplex and $L = -L$, then $d(K,L) = n$. The following question arises naturally: Is equality only attained when one of the sets is a simplex? Leichtweiss, and later Palmon proved that if $d(K, B_2^n) = n$, where B_2^n is the Euclidean ball, then K is the simplex. We proved the affirmative answer to the question in the case when one of the bodies is strictly convex or smooth, thus obtaining a generalization of the result of Leichtweiss and Palmon. Joint work with Carlos Hugo Jiménez, Universidad de Sevilla, Spain.

Carlos M. Nicolas. University of North Carolina, Greensboro, NC.

Interval decompositions of k-edges and applications.

Given a set S of points in the plane, a *k-edge interval* is the set of k-edges of S whose normal vectors belong to a given interval of the unit circle. These intervals are complete in the following sense: any k-edge interval is equal to an i-edge interval for the set of vertices incident to its edges, for some $i \leq k$. This provides a recursive approach to the study of k-edges because k-edge intervals decompose into simpler edge-disjoint sub-intervals. Using this approach we obtain alternative proofs, sometimes simpler, for several properties of the set of k-edges such as the k-edge crossing identity and the current lower bound on the number of k-edges.

Deborah Oliveros. Universidad Nacional Autonoma de Mexico, Mexico City, Mexico.

Helly-type theorems and intersection graphs: one nice relation.

Given a family of convex sets in the n-dimensional Euclidean space, it is natural to define a graph called the intersection graph, were its vertices are the elements of the family and two vertices will have an edge in common if they have a point in common, a similar definition can be done to define uniform intersection λ-hypergraphs, were λ vertices become a hyperedge if the corresponding λ convex sets intersect. It turns out, that there are several Helly type theorems that can be investigated by using chromatic number and extremal theory such as covering or transversal numbers for intersection graphs and hypergraphs. In this talk, we will discuss some of these applications.

Egon Schulte. Northeastern University, Boston, MA.

Regular polyhedra of index two in space.

A polyhedron in Euclidean 3-space is said to be a regular polyhedron of index 2 if it is combinatorially regular but "fails geometric regularity by a factor of 2"; that is, its combinatorial automorphism group is flag-transitive but its geometric symmetry group has two flag orbits. We report on the complete classification of the regular polyhedra of index 2. This is joint work with Anthony Cutler and is described in his 2009 PhD thesis at Northeastern University.

Valeriu Soltan. George Mason University, Fairfax, VA.

Convex surfaces with planar quadric sections.

A convex quadric surface in $\mathbb{R}^n$ is the boundary of a convex component of $\mathbb{R}^n \setminus Q$, if any, where Q is a real quadric surface. We review existing results and open problems related to characterizations of convex quadric surfaces in terms of their planar sections.

Csaba D. Tóth. University of Calgary, Calgary, Alberta, Canada.

Convex subdivisions for point sets.

A *convex subdivision* for a set S of n points in the plane is a planar straight line graph such that its vertex set contains S, the bounded faces are convex, and the outer face is the complement of the convex hull of S. This talk will survey recent extremal results for convex subdivisons and some applications. The results include upper and lower bounds for the minimum number of faces (with or without Steiner points), minimum total edge length, and the minimum size of a convex subdivision contained in a triangulation of n points in the plane.

Gabor A. Tóth. Rutgers University, Camden, NJ.

A measure of symmetry for convex sets and its application to moduli for minimal immersions of spheres.

Asymmetry of a compact convex body $L \subset \mathbb{R}^n$ viewed from an interior point O can be measured by considering how far L is from its inscribed simplices that contain O. This leads to a sequence of measures of symmetry $\{\sigma_k(L, O)\}_{k \geq 1}$ in the sense of Grünbaum. This sequence of measures has interesting arithmetic properties. The interior of L naturally splits into regular and singular sets, where the singular set consist of points with largest possible $\sigma_n(L, O)$. In general, to calculate the regular and singular sets is difficult. In this talk we give a variety of methods that facilitate this calculation. The methods are illustrated by several examples. The original motivation for introducing these measures is to describe the geometry of the DoCarmo-Wallach moduli spaces of minimal immersions of spheres. We use the DeTurck-Ziller minimal orbit method for $SU(2)$ to calculate these measures on the $SU(2)$-equivariant moduli of S^3 into spheres.

Rick Vitale. University of Connecticut, Storrs, CT.

Intrinsic volumes.

Intrinsic volumes are key functionals defined on convex bodies and measure in a number of different questions. Here we discuss in particular their connections with Gaussian processes, with the Wills functional, and with some open problems, as time permits.

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