Convexity and Combinatorics

AMS Special Session

November 6-7, 2004, Pittsburgh, Pennsylvania

Organizers: Jim Lawrence and Valeriu Soltan, George Mason University, Virginia

PROGRAM OF THE SESSION

Saturday, November 6

8:00 am	M. Bayer. *Reconstruction of polytopes as Eulerian posets.*
8:30 am	W. Finbow-Singh. *Simplicial neighborly 5-polytopes with nine vertices.*
9:00 am	C. Lee,* M. Menzel, L. Schmidt. *Some construction techniques for convex polytopes.*
9:30 am	W. Morris,* B. Gaertner, L. Ruest. *A generalization of the Holt-Klee theorem.*
10:00am	J. Lawrence. *Multiplication in the polytope groups.*
10:30am	Z. Mustafaev. *On the perimeter and area of the unit disc in Minkowski planes.*
3:00 pm	H. Martini. *Some new results on geometric graphs.*
3:30 pm	M. Breen. *Helly-type theorems for intersections of starshaped sets.*
4:00 pm	T. Bisztriczky,* F. Fodor, D. Oliveros. *Large transversals to small families of unit disks.*
4:30 pm	A. Dumitrescu. *On distinct distances from a vertex of a convex polygon.*
5:00 pm	D. Ismailescu. *A dense planar set from iterated line intersections.*
5:30 pm	N. Elkies, L. Pretorius, K. Swanepoel.* *Sylvester-Gallai theorems for complex numbers and quaternions.*

Sunday, November 7

8:00 am	W. Kuperberg. *On the space of affine classes of convex bodies.*
8:30 am	D. Larman,* G. Sojka. *Determining a convex body by minor subsets of the boundary.*
9:00 am	P. Brass. *On Lebesgue's universal cover problem.*
9:30 am	A. Bezdek. *On the number of mutually touching cylinders.*
10:00am	G. Fejes Tóth. *Covering with fat convex disks.*
3:00 pm	R. Dawson. *Tilings of the sphere with congruent spherical triangles.*
3:30 pm	K. Bezdek, M. Naszodi, D. Oliveros.* *Antipodality in hyperbolic space.*
4:00 pm	B. Dekster. *An angle in Minkowski space.*
4:30 pm	V. Soltan. *Pairs of convex bodies with centrally symmetric intersections of translates.*

ABSTRACTS

Margaret M. Bayer, University of Kansas, Lawrence, KS. *Reconstruction of polytopes as Eulerian posets.*

Results on combinatorial reconstruction for polytopes are of the following form: If P and Q are convex d-polytopes, P is in a specified class (e.g., simplicial polytopes, simple polytopes, zonotopes), and the k-skeletons of P and Q are combinatorially equivalent, then P and Q are combinatorially equivalent. (The k-skeleton is the subcomplex of the boundary complex consisting of all faces of dimension at most k.) In this talk we consider what happens if we relax the hypothesis on Q, requiring only that Q be an Eulerian partially ordered set. We show that if P is a simplicial d-polytope, then the face lattice of P is the unique Eulerian poset agreeing with P on all but the dimension r faces of P, for $0 \leq r \leq d-2$, and give a counterexample for $r = d-1$

Andras Bezdek, Auburn University, Auburn, AL. *On the number of mutually touching cylinders.*

The following problem was posed by Littlewood in 1968: What is the maximum number of congruent infinite circular cylinders that can be arranged in $\mathbb{E}^3$ so that any two of them are touching? Is it 7? This problem is still open. The analogous problem concerning circular cylinders of infinite length became known as a mathematical puzzle due to a popular book of Gardner. Find an arrangement of 7 cigarettes so that any two of them touch each other. The question whether 7 is the largest such number is open. A very large bound concerning the above question on infinite circular cylinders, expressed in terms of various Ramsey constants was found by the author in 1991. The bound was so large that it merely showed the existence of a finite bound. In this talk we use a different approach to show that at most 24 congruent infinite circular cylinders can be arranged so that any two of them are touching. We also address the problem when the cylinders are not necessarily congruent.

Karoly Bezdek, Marton Naszodi, Déborah Oliveros*, University of Calgary, Calgary, Alberta, Canada. *Antipodality in hyperbolic space.*

A set of points in the Euclidean d-space is called antipodal if through every pair of points in the set, there is a pair of parallel hyperplanes supporting the set. According to a well known result of Danzer and Grünbaum, conjectured independently by Erdős and Klee, the cardinality of any antipodal set in $\mathbb{E}^d$ is at most 2^d. Some other bounds have been found in lower dimensions. We will discuss some various possible ways to define hyperbolic antipodality and present similar results to the ones above on the cardinality of an antipodal set in the hyperbolic d-space.

Tibor Bisztriczky*, Ferenc Fodor, Déborah Oliveros, University of Calgary, Calgary, Alberta, Canada. *Large transversals to small families of unit disks.*

We determine conditions under which a finite family $\mathcal{F}$ of disjoint unit disks has a transversal line that intersects all but at most one member of the family. This problem is closely related to the Katchalski-Lewis Conjecture for plane convex sets. We show that $T(4)$ implies T-1 if $\mathcal{F}$ has at most seven elements.

Peter Brass, City College of New York, New York, NY. *On Lebesgue's universal cover problem.*

The universal cover problem in its most classical version asks for the minimum area of a convex set in the plane that contains congruent copies of any set of diameter 1. This problem is ascribed to Lebesgue, and was first studied in a paper by J. Pál in 1920. He gave a lower bound of 0.825 and constructed a universal cover of area 0.845. Since then a number of further covers have been constructed, slightly reducing the upper bound to 0.844, but the lower bound stood untouched for the next seventy years, until in 1994 G. Elekes increased it to 0.827. In this talk I present a further improvement of the lower bound to 0.83, using a combination of geometric and computational methods.

Marilyn Breen, University of Oklahoma, Norman, OK. *Helly-type theorems for intersections of starshaped sets.*

Some familiar results for intersections of convex sets may be extended to intersections of starshaped sets. Among the results are the following: Let k and d be fixed integers, $0 \leq k \leq d$, and let $\mathcal{F}$ be a collection of sets in $\mathbb{E}^d$. If every countable subfamily of $\mathcal{F}$ has a starshaped intersection, then $\cap\{S : S \in \mathcal{F}\}$ is (nonempty and) starshaped as well. If every countable subfamily of $\mathcal{F}$ has as its intersection a starshaped set whose kernel is at least k-dimensional, then the kernel of $\cap\{S : S \in \mathcal{F}\}$ is at least k-dimensional, too.

Robert J. MacG. Dawson, Saint Mary's University, Halifax, Nova Scotia, Canada. *Tilings of the sphere with congruent spherical triangles.*

Sommerville, in the 1920's, gave a characterization of the ways in which a sphere could be tiled with congruent spherical triangles, meeting edge-to-edge and satisfying some additional assumptions. Davies, in the 1960's, dropped some of the assumptions but retained the assumption of edge-to-edge contact; he obtained thereby a larger set of tilings, but left a lot of details to the reader. Ueno and Agaoka, in 2001, filled in the details but greatly increased the length of the classification.

In the last few years, Blair Doyle and the speaker have been working on the classification of tilings of the sphere with congruent triangles, dropping the assumption of edge-to-edge contact. The tilings tend to be rather elegant and complex, and tend to have a chiral symmetry, whereas edge-to-edge tilings usually have mirror symmetries. The isosceles and right-angled cases have been completely classified, and significant progress has been made towards a complete classification. This paper will survey the results obtained so far.

Boris V. Dekster, Mount Allison University, New Brunswick, Canada. *An angle in Minkowski space.*

A new angular measure in Minkowski space is introduced. It is defined for a cone of any dimension, is additive and invariant under affine transformations. For the dimension 2, it has a clear interpretation as an "amount of rotation" from one direction to another one.

Adrian Dumitrescu, University of Wisconsin-Milwaukee, Milwaukee, WI. *On distinct distances from a vertex of a convex polygon.*

Given a set P of n points in convex position in the plane, we prove that there exists a point $p \in P$ such that the number of distinct distances from p is at least $\lceil (13n-6)/36 \rceil$. The best previous bound, $\lceil n/3 \rceil$, from 1952, is due to Leo Moser.

Noam Elkies, Lou M. Pretorius, Konrad J. Swanepoel*, University of South Africa, Pretoria, South Africa. *Sylvester-Gallai theorems for complex numbers and quaternions.*

A Sylvester-Gallai (SG) configuration is a finite set S of points such that the line through any two points in S contains a third point of S. According to the Sylvester-Gallai Theorem, an SG configuration in real projective space must be collinear. A problem of Serre (1966) asks whether an SG configuration in a complex projective space must be coplanar. This was proved by Kelly (1986) using a deep inequality of Hirzebruch. We give an elementary proof of this result, and then extend it to show that an SG configuration in projective space over the quaternions must be contained in a three-dimensional flat.

Gábor Fejes Tóth, Alfréd Rényi Mathematical Institute, Budapest, Hungary. *Covering with fat convex disks.*

According to a theorem of L. Fejes Tóth, if non-crossing congruent copies of a convex disc K cover a convex hexagon H, then the density of the discs relative to H is at least area $K/f_K(6)$ where $f_K(6)$ denotes the maximum area of a hexagon contained in K. Two convex discs cross if removing their intersection from them each disc becomes non-connected. The assumption that the discs do not cross seems to be superfluous and it has been an open problem for over 50 years to get rid of this assumption. We say that a convex disc K is *r-fat* if it is contained in a unit circle C and contains a concentric circle c of radius r. Recently, Heppes showed that the above inequality holds without the non-crossing assumption if K is an 0.8561-fat ellipse. We show that the non-crossing assumption can be omitted if K is an r_0-fat convex disc with $r_0 = 0.933$ or an r_1-fat ellipse with $r_1 = 0.741$.

Wendy A. Finbow-Singh, Acadia University, Nova Scotia, Canada. *Simplicial neighbourly 5-polytopes with nine vertices.*

Among the d-polytopes with v vertices, the neighbourly polytopes have the greatest number of facets. This maximum property of neighbourly polytopes has prompted researchers to compose lists of them. In this talk, we will discuss the simplicial, neighbourly 5-polytopes with nine vertices. We show that there are at least one hundred, twenty-six of them. We discuss the connection between the neighbourly 4-polytopes with eight vertices, the neighbourly 5-polytopes with nine vertices, and the neighbourly 6-polytopes with ten vertices.

Dan Ismailescu, Hofstra University, Hempstead, NY. *A dense planar set from iterated line intersections.*

Given S_1, a set of points in the plane, not all on a line, we define a sequence of planar point sets $\{S_i\}_{i=1}^{\infty}$ as follows: with S_i already determined, let L_i be the set of all the lines determined by pairs of points from S_i, and let S_{i+1} be the set of all the intersection points of lines in L_i. We show that with the exception of some very particular starting configurations, the limiting point set $\cup_{i=1}^{\infty} S_i$ is everywhere dense in the plane. This is joint work with Radoš Radoičić.

Wlodzimierz Kuperberg, Auburn University, Auburn, AL. *On the space of affine classes of convex bodies.*

Let $\mathcal{K}_n$ denote the space of convex bodies in $\mathbb{E}^n$ with topology generated by the Hausdorff metric, and let $\mathcal{K}_n^*$ be the quotient space of $\mathcal{K}_n$ obtained by identifying convex bodies that are affinely equivalent. It is known that $\mathcal{K}_n^*$ is metrizable and compact [A.M. Macbeath, 1951]. We establish some other topological properties of $\mathcal{K}_n^*$, such as contractibility and local connectedness. We are particularly interested in the special transformations $T^* : \mathcal{K}_n^* \to \mathcal{K}_n^*$ that are induced by affine-equivariant maps $T : \mathcal{K}_n \to \mathcal{K}_n$. For example, $\mathcal{K}_n^*$ is affine-equivariantly contractible to a unique point, and every affinely-equivariant contraction of $\mathcal{K}_n$ defines a starlike structure on $\mathcal{K}_n^*$ with a unique star-center, the ellipsoid class.

David Larman, University College London, London, UK. *Determining a convex body by minor subsets of the boundary.*

For a fixed point x in the interior of an n-dimensional convex body K consider the set $K(x)$ of all those points y on the boundary of K which are closer to x than is the other boundary point lying on the line through x and y. Surprisingly, $K(x)$ can be most of $\operatorname{bd} K$ but $\operatorname{bd} K$ cannot be covered by fewer than $n+1$ sets of the form $\operatorname{cl} K(x)$.

Jim Lawrence, George Mason University, Fairfax, VA. *Multiplication in the polytope groups.*

There are several ways to introduce binary operations on the additive group of simple functions generated by indicator functions of polyhedra in $\mathbb{E}^d$, giving the group the structure of a commutative ring. Some of these rings carry a strong geometrical flavor. As evidence of this we consider several, sometimes well-known, geometrical decompositions which arise from identities in the rings.

Carl W. Lee*, Matt Menzel, Laura Schmidt, University of Kentucky, Lexington, KY. *Some construction techniques for convex polytopes.*

We will discuss some construction techniques for convex polyhedra, with an eye toward realizing some classes of f-vectors and flag f-vectors.

1. Billera and Lee describe a set of necessary conditions for f-vectors of antistars in simplicial polytopes, and hence for regular triangulations and (by duality) for unbounded, simple polyhedra. It is not yet known whether these conditions are sufficient. In joint work with Laura Schmidt we construct certain classes of regular triangulations to demonstrate

the sufficiency of these conditions in low dimensions. The construction exploits some of the combinatorial structure of the simplicial polytopes used in the proof of the g-Theorem.

2. The set of flag f-vectors of four-dimensional polytopes has not yet been characterized. Extending the sewing technique of Shemer (dual to Barnette's facet-splitting), in joint work with Matt Menzel we discuss a generalized sewing method to construct non-simplicial polytopes. One encouraging feature of this method is that it encompasses the construction of ordinary polytopes by Bisztriczky and Dinh. We present some results on the set of flag f-vectors of four-dimensional polytopes achievable by this process.

Horst Martini, University of Technology Chemnitz, Chemnitz, Germany. *Some new results on geometric graphs.*

A finite set E of line segments in the plane $\mathbb{E}^2$ may be considered as a *geometric graph* $G = (V, E)$, where V denotes the set of vertices of the segments from E and no open segment from E contains a vertex from V. We present some new results on applications of certain types of geometric graphs in geometric convexity, e.g. referring to curves or bodies of constant width, to the number of maximal regular simplices determined by m points in $\mathbb{E}^n$, and to special convex bodies in close relation to the famous Borsuk problem from the combinatorial geometry of convex bodies.

Walter D Morris*, Bernd Gaertner, Leo Ruest, George Mason University, Fairfax, VA. *A generalization of the Holt-Klee theorem.*

The Holt-Klee theorem is a directed version of Balinski's d-connectivity theorem for convex polytopes. It states that if the edges of a d-polytope P are oriented consistently with the direction of increase of a linear function on P, then there are d disjoint directed paths from the source to the sink. We show that an analogous theorem holds for dual graphs of complete pointed simplicial fans. This leads to a new combinatorial necessary condition for an orientation of the d-cube graph to be induced by a linear function on a polytope combinatorially equivalent to the d-cube.

Zokhrab Mustafaev, Ithaca College, Ithaca, NY. *On the perimeter and area of the unit disc in Minkowski planes.*

In this talk we show that the ratio of the Minkowski length of the unit 'circle' to the Holmes-Thompson area of the unit disc is greater than or equal to 2, and that the ratio 2 is achieved only for Minkowski planes which are affine equivalent to the Euclidean plane. In other words, the ratio is 2 only when the unit 'circle' is an ellipse.

Valeriu Soltan, George Mason University, Fairfax, VA. *Pairs of convex bodies with centrally symmetric intersections of translates.*

For a pair of convex bodies K and K' in $\mathbb{E}^n$, the n-dimensional intersections $K \cap (x + K')$, $x \in \mathbb{E}^n$, are centrally symmetric if and only if K and K' are represented as direct sums $K = R \oplus P$ and $K' = R' \oplus P'$ such that: (i) R is a closed convex set of some dimension m, $0 \leq m \leq n$, and $R' = z - R$ for a suitable vector $z \in \mathbb{E}^n$, (ii) P and P' are isothetic parallelotopes, both of dimension $n - m$.

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