

7.2. Quadratic Forms

Definition. A quadratic form on \mathbb{R}^n is a function of the form $Q(\bar{x}) = \bar{x}^T A \bar{x}$, where \bar{x} is a variable vector in \mathbb{R}^n and A is an $n \times n$ symmetric matrix.

The simplest example of a nonzero quadratic form is $Q(\bar{x}) = \bar{x}^T I \bar{x} = \|\bar{x}\|^2$.

Example 1. Compute $Q(\bar{x}) = \bar{x}^T A \bar{x}$, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$.

Solution. $\bar{x}^T A \bar{x} = [x_1 \ x_2] \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= [x_1 \ x_2] \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix} = x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2.$$

Example 2. Write the function

$$Q(\bar{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$$

as $\bar{x}^T A \bar{x}$.

Solution. Expressing $-x_1x_2 + 8x_2x_3$ as

$$-\frac{1}{2}x_1x_2 - \frac{1}{2}x_2x_1 + 4x_2x_3 + 4x_3x_2, \text{ one has}$$

$$Q(\bar{x}) = [x_1 \ x_2 \ x_3] \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{x}^T A \bar{x}.$$

Change of Variable in a Quadratic Form

Given a variable vector \bar{x} in \mathbb{R}^n and an invertible $n \times n$ matrix P , the equation $\bar{x} = P\bar{y}$ is called a change of variable.

If the change of variable $\bar{x} = P\bar{y}$ is made in a quadratic form $\bar{x}^T A \bar{x}$, then

$$\bar{x}^T A \bar{x} = (P\bar{y})^T A (P\bar{y}) = \bar{y}^T (P^T A P) \bar{y}.$$

The matrix $P^T A P$ is symmetric:

$$(P^T A P)^T = P^T A^T P^{TT} = P^T A P.$$

So, the function $\bar{y}^T D \bar{y}$, where $D = P^T A P$ is a quadratic form.

Example 4. Make a change of variable that nullifies the term $-8x_1 x_2$ in $Q(\bar{x}) = x_1^2 - 8x_1 x_2 - 5x_2^2$.

Solution. First, we observe that

$$Q(\bar{x}) = [x_1 \ x_2] \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Next, we orthogonally diagonalize A (see Section 7.1).

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda) - 16 \\ &= \lambda^2 + 4\lambda - 1 = (\lambda-3)(\lambda+7). \end{aligned}$$

Solving the equation $(A - 3I)\bar{x} = \bar{0}$, we obtain

$$\bar{x} = \begin{bmatrix} 2x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \quad \text{Let } \bar{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}.$$

Solving the equation $(A + 7I)\bar{x} = \bar{0}$, we get

$$\bar{x} = \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \quad \text{Let } \bar{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

$$\text{Put } P = \begin{bmatrix} 2\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}.$$

$$\text{Then } A = PDP^T \text{ and } D = P^TAP.$$

$$\text{Let } \bar{x} = P\bar{y}, \text{ where } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

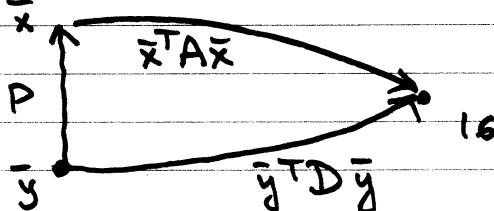
$$\begin{aligned} \text{Then } x_1^2 - 8x_1x_2 - 5x_2^2 &= \bar{x}^T A \bar{x} = (P\bar{y})^T A (P\bar{y}) \\ &= \bar{y}^T (P^T A P) \bar{y} = \bar{y}^T D \bar{y} \\ &= [y_1, y_2] \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3y_1^2 - 7y_2^2. \end{aligned}$$

For instance, if $\bar{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, then

$$\bar{y} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}.$$

$$Q(\bar{x}) = Q\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = 2^2 - 8 \cdot 2(-2) - 5(-2)^2 = \underline{16}$$

$$\bar{y}^T D \bar{y} = 3\left(\frac{6}{\sqrt{5}}\right)^2 - 7\left(-\frac{2}{\sqrt{5}}\right)^2 = 3\left(\frac{36}{5}\right) - 7\left(\frac{4}{5}\right) = \underline{16}.$$



Theorem 4. Let A be an $n \times n$ symmetric matrix.

Then there is an orthogonal change of variable,

$\bar{x} = P\bar{y}$, that transforms the quadratic form

$\bar{x}^T A \bar{x}$ into a quadratic form $\bar{y}^T D \bar{y}$, where

D is a diagonal $n \times n$ matrix whose diagonal entries are the eigenvalues of A .