

## 7.1 Diagonalization of Symmetric Matrices

Definition. A symmetric matrix is a matrix  $A$  such that  $A^T = A$ .

Note. If a symmetric matrix  $A$  has size  $m \times n$ , then  $n \times m$  is the size of  $A^T$ . The equality  $A^T = A$  implies that  $m = n$ . So, a symmetric matrix must be square, of size  $n \times n$ .

Example 1. Symmetric matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix}$$

Theorem 1. If  $A$  is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal:

If  $A\bar{v}_1 = \lambda_1 \bar{v}_1$  and  $A\bar{v}_2 = \lambda_2 \bar{v}_2$ , where  $\lambda_1 \neq \lambda_2$ ,

then  $\bar{v}_1 \cdot \bar{v}_2 = 0$ .

Definition. A matrix  $P$  of size  $n \times n$  is called orthogonal if  $P^{-1} = P^T$ .

An  $n \times n$  matrix  $A$  is said to be orthogonally diagonalizable if there is an orthogonal matrix  $P$  and a diagonal matrix  $D$  such that

$$A = PDP^T = PDP^{-1}.$$

Theorem 2. An  $n \times n$  matrix  $A$  is orthogonally diagonalizable if and only if  $A$  is symmetric.

For instance, if  $A$  is orthogonally diagonalizable as  $A = PDP^T$ , then

$$A^T = (PDP^T)^T = P^T D P^T = PDP^T = A.$$

Example 3. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

whose characteristic equation is

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2).$$

Solution. Solving the matrix equations

$$(A - 7I)\bar{x} = \bar{0} \quad \text{and} \quad (A + 2I)\bar{x} = \bar{0},$$

(3)

we obtain the following eigenvectors:

$$\text{for } \lambda=7: \bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix}; \text{ for } \lambda=-2: \bar{v}_3 = \begin{bmatrix} -1 \\ -0.5 \\ 1 \end{bmatrix}.$$

Although  $\bar{v}_1$  and  $\bar{v}_2$  are linearly independent, they are not orthogonal:  $\bar{v}_1 \cdot \bar{v}_2 = 1(-0.5) + 0 \cdot 1 + 1 \cdot 0 = -0.5 \neq 0$ .

Using the projection of  $\bar{v}_2$  onto  $\bar{v}_1$ , we obtain an eigenvector

$$\bar{e}_2 = \bar{v}_2 - \frac{\bar{v}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 = \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix} - \frac{-0.5}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1 \\ 0.25 \end{bmatrix}.$$

Normalizing  $\bar{v}_1, \bar{e}_2, \bar{v}_3$ , we obtain the unit vectors

$$\bar{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

Let

$$P = [\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

we have  $A = PDP^{-1} = PD\bar{P}^T$ .

## The Spectral Theorem

The set of eigenvalues of a matrix  $A$  is sometimes called the spectrum of  $A$ .

Theorem 3. An  $n \times n$  symmetric matrix  $A$  has the following properties.

- a.  $A$  has  $n$  real eigenvalues, counting multiplicities.
- b. The dimension of the eigenspace for each eigenvalue  $\lambda$  equals the multiplicity of  $\lambda$ .
- c. The eigenspaces are mutually orthogonal.
- d.  $A$  is orthogonally diagonalizable.

## Spectral Decomposition

Suppose that a symmetric  $n \times n$  matrix  $A$  is expressed as  $A = P D P^T$ , where  $P = [\bar{u}_1 \dots \bar{u}_n]$  is formed by orthogonal eigenvectors  $\bar{u}_1, \dots, \bar{u}_n$  which correspond to eigenvalues  $\lambda_1, \dots, \lambda_n$ . Then

$$A = P D P^T = [\bar{u}_1 \dots \bar{u}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{u}_1^T \\ \vdots \\ \bar{u}_n^T \end{bmatrix}$$

$$= [\lambda_1 \bar{u}_1 \dots \lambda_n \bar{u}_n] \begin{bmatrix} \bar{u}_1^T \\ \vdots \\ \bar{u}_n^T \end{bmatrix} = \lambda_1 \bar{u}_1 \bar{u}_1^T + \dots + \lambda_n \bar{u}_n \bar{u}_n^T.$$

This representation of  $A$  is called a spectral decomposition of  $A$ . Each term  $\lambda_i \bar{u}_i u_i^\top$  is an  $n \times n$  matrix of rank 1 since every column of  $\lambda_i \bar{u}_i u_i^\top$  is a multiple of  $\bar{u}_i$ .

Example 4. Construct a spectral decomposition of the matrix  $A$  that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

Solution. Here  $\lambda_1 = 8$ ,  $\lambda_2 = 3$ , and the eigenvectors are

$$\bar{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

$$\bar{u}_1 \bar{u}_1^\top = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

$$\bar{u}_2 \bar{u}_2^\top = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

Furthermore,

$$8 \bar{u}_1 \bar{u}_1^\top + 3 \bar{u}_2 \bar{u}_2^\top = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A.$$