6.4. The Gram-Schmidt Process

<u>Theorem</u> (The Gram-Semidt Process)

Given a basis $\{\bar{x}_1, \dots, \bar{x}_p\}$ for a subspace W of R^n, define

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2} \bar{v}_2$$

$$\cdots\cdots$$

$$\bar{v}_p = \bar{x}_p - \frac{\bar{x}_p \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1 - \cdots - \frac{\bar{x}_p \cdot \bar{v}_{p-1}}{\bar{v}_{p-1} \cdot \bar{v}_{p-1}} \bar{v}_{p-1}.$$

Then $\{\bar{v}_1, \dots, \bar{v}_p\}$ is an orthogonal basis for W such that $\operatorname{Span}\{\bar{v}_1, \dots, \bar{v}_k\} = \operatorname{Span}\{\bar{x}_1, \dots, \bar{x}_k\}$ for $1 \le k \le p$.

<u>Example 1.</u> Let $W = \operatorname{span}\{\bar{x}_1, \bar{x}_2\}$, where $\bar{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$, $\bar{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{\bar{v}_1, \bar{v}_2\}$ for W.

<u>Solution</u>. Let $\bar{v}_1 = \bar{x}_1$ and $\bar{p}$ be the projection of $\bar{x}_2$ on $\bar{x}_1$. Put

$$\bar{v}_2 = \bar{x}_2 - \bar{p} = x_2 - \frac{\bar{x}_2 \cdot \bar{x}_1}{\bar{x}_1 \cdot \bar{x}_1} \bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Then $\{\bar{v}_1, \bar{v}_2\}$ is an orthogonal basis for W.

Example 2. Construct an orthogonal basis for $W = \text{Span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ where

$$\bar{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

Solution. Let $\bar{v}_1 = \bar{x}_1$, $W_1 = \text{Span}\{\bar{x}_1\}$

$$\bar{v}_2 = \bar{x}_2 - \text{proj}_{W_1}\bar{x}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix}.$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2 = \begin{bmatrix}0\\-2/3\\2/3\\2/3\end{bmatrix}.$$

So, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthogonal basis for W.

Orthonormal Bases

An orthonormal basis is constracted easily from an orthogonal basis $\{\bar{v}_1, \dots, \bar{v}_p\}$ by normalizing all of $\bar{v}_1, \dots, \bar{v}_p$.

Example 3. Example 1 constructed the orthogonal basis $\bar{v}_1 = \begin{bmatrix}3\\6\\0\end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix}0\\0\\2\end{bmatrix}$.

An orthonormal basis is

$$\bar{u}_1 = \frac{1}{\|\bar{v}_1\|}\bar{v}_1 = \frac{1}{\sqrt{45}}\begin{bmatrix}3\\6\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}, \quad \bar{u}_2 = \frac{1}{\|\bar{v}_2\|}\bar{v}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$