6.3. Orthogonal Projections

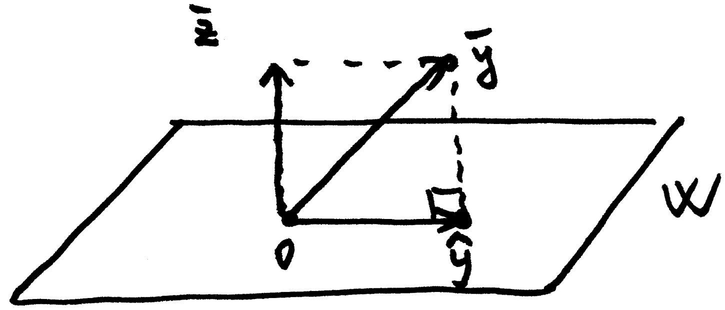
Theorem. Let W be a subspace of R^n. Then each $\bar{y}$ in R^n can be written uniquely in the form

$$\bar{y} = \hat{y} + \bar{z} \qquad (1)$$

where $\hat{y}$ is in W and $\bar{z}$ is in $W^\perp$. In fact, if $\{\bar{u}_1, \dots, \bar{u}_p\}$ is any orthogonal basis for W, then

$$\hat{y} = \frac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\bar{u}_1 + \dots + \frac{\bar{y}\cdot\bar{u}_p}{\bar{u}_p\cdot\bar{u}_p}\bar{u}_p \qquad (2)$$

and $\bar{z} = \bar{y} - \hat{y}$.



The vector $\hat{y}$ in (1) is called the orthogonal projection of $\bar{y}$ onto W. When W is a one-dimensional subspace spanned by a vector $\bar{u}$, formula (2) matches the formula given in Section 6.2.

Example. Let $\bar{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\bar{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Observe that $\{\bar{u}_1, \bar{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$. Write $\bar{y}$ as the sum of a vector in W and a vector orthogonal to W.

Solution. The orthogonal projection of $\bar{y}$ onto W is

$$\hat{y} = \frac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\,\bar{u}_1 + \frac{\bar{y}\cdot\bar{u}_2}{\bar{u}_2\cdot\bar{u}_2}\,\bar{u}_2$$

$$= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{15}{30}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

$$\bar{z} = \bar{y} - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

Properties of Orthogonal Projections

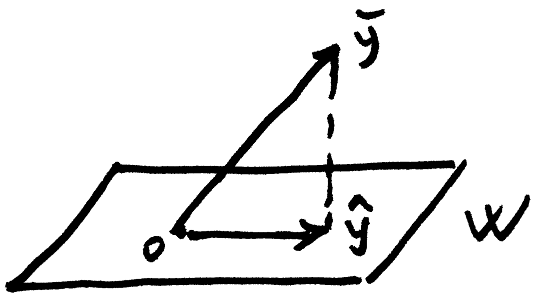
Theorem (The Best Approximation Theorem)

Let W be a subspace of R^n, $\bar{y}$ any vector in R^n, and $\hat{y}$ the orthogonal projection of $\bar{y}$ onto W. Then $\hat{y}$ is the closest point in W to $\bar{y}$, in the sense that

$$\| y - \hat{y} \| < \| \bar{y} - \bar{v} \|$$

for all $\bar{v}$ in W distinct from $\hat{y}$.

The distance from a point $\bar{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\bar{y}$ to the nearest point in W.



<u>Example</u>. Find the distance from $\bar{y}$ to $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$, where

$$\bar{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \bar{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

<u>Solution</u>. By the Best Approximation Theorem, the distance from $\bar{y}$ to W is $\|\bar{y} - \hat{y}\|$, where $\hat{y} = \text{proj}_W \bar{y}$. Since $\{\bar{u}_1, \bar{u}_2\}$ is an orthogonal basis for W,

$$\hat{y} = \frac{15}{30}\bar{u}_1 + \frac{-21}{6}\bar{u}_2 = \frac{1}{2}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}.$$

$$\bar{y} - \hat{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \quad \|\bar{y} - \hat{y}\|^2 = 3^2 + 6^2 = 45.$$

So, the distance from $\bar{y}$ to W is $\sqrt{45} = 3\sqrt{5}$.

<u>Theorem.</u> If $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then for any vector $\bar{y}$ in $\mathbb{R}^n$,

$$\text{proj}_W \bar{y} = (\bar{y} \cdot \bar{u}_1)\bar{u}_1 + \cdots + (\bar{y} \cdot \bar{u}_p)\bar{u}_p .$$

Furthermore, if $U = [\bar{u}_1 \ldots \bar{u}_p]$, then

$$\text{proj}_W \bar{y} = U U^T \bar{y} .$$

<u>Example.</u> Let $\bar{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\bar{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, and $W = \text{span}\{\bar{u}_1\}$.

Compute $\text{proj}_W \bar{y}$ and $U U^T \bar{y}$.

<u>Solution.</u> $\text{proj}_W \bar{y} = (\bar{y} \cdot \bar{u}_1)\bar{u}_1 = \left(7/\sqrt{10} - 27/\sqrt{10}\right) \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$$= \begin{bmatrix} -20/10 \\ 60/10 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

$$U U^T = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$$

$$U U^T \bar{y} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -20/10 \\ 60/10 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

So, $\text{proj}_W \bar{y} = U U^T \bar{y}$.
