

6.2 Orthogonal Sets

Definition. A set of vectors $\{\bar{u}_1, \dots, \bar{u}_p\}$ in \mathbb{R}^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\bar{u}_i \cdot \bar{u}_j = 0$ whenever $i \neq j$.

Example. Show that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal set, where $\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\bar{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$.

Solution. $\bar{u}_1 \cdot \bar{u}_2 = 3(-1) + 1(2) + 1(1) = 0$,
 $\bar{u}_1 \cdot \bar{u}_3 = 3\left(-\frac{1}{2}\right) + 1(-2) + 1\left(\frac{7}{2}\right) = 0$,
 $\bar{u}_2 \cdot \bar{u}_3 = (-1)\left(-\frac{1}{2}\right) + 2(-2) + 1\left(\frac{7}{2}\right) = 0$.

Theorem. If $S = \{\bar{u}_1, \dots, \bar{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent.

Definition. An orthogonal basis for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.

Example. The standard basis $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $\bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ is an orthogonal basis for \mathbb{R}^n .

Theorem. Let $\{\bar{u}_1, \dots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of R^n . For each \bar{y} in W , given by

$$\bar{y} = c_1 \bar{u}_1 + \dots + c_p \bar{u}_p,$$

we have

$$c_i = \frac{\bar{y} \cdot \bar{u}_i}{\bar{u}_i \cdot \bar{u}_i}, \quad i = 1, \dots, p.$$

Example. Express the vector $\bar{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Solution. We know that $\bar{u}_1 \cdot \bar{u}_1 = 11$, $\bar{u}_2 \cdot \bar{u}_2 = 6$, $\bar{u}_3 \cdot \bar{u}_3 = 33/2$.

$$\text{Furthermore, } \bar{y} \cdot \bar{u}_1 = 6(3) + 1(1) + (-8)(1) = 11,$$

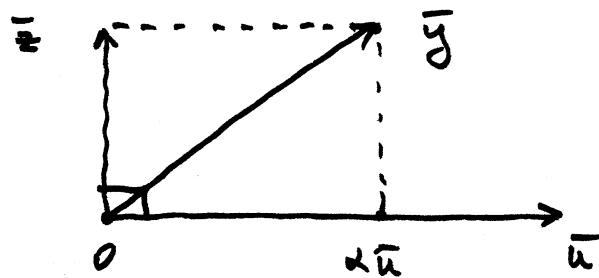
$$\bar{y} \cdot \bar{u}_2 = 6(-1) + 1(2) + (-8)(1) = -12,$$

$$\bar{y} \cdot \bar{u}_3 = 6(-\frac{1}{2}) + 1(-2) + (-8)(\frac{7}{2}) = -33.$$

$$\begin{aligned} \text{So, } \bar{y} &= \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} \bar{u}_2 + \frac{\bar{y} \cdot \bar{u}_3}{\bar{u}_3 \cdot \bar{u}_3} \bar{u}_3 \\ &= \frac{11}{11} \bar{u}_1 + \frac{-12}{6} \bar{u}_2 + \frac{-33}{33/2} \bar{u}_3 \\ &= \bar{u}_1 - 2\bar{u}_2 - 2\bar{u}_3. \end{aligned}$$

An Orthogonal Projection onto a Vector

Given a nonzero vector \bar{u} in \mathbb{R}^n and a vector \bar{y} in \mathbb{R}^n , we wish to write $\bar{y} = \hat{y} + \bar{z}$, where $\hat{y} = \alpha\bar{u}$ for some scalar α and \bar{z} is a vector orthogonal to \bar{u} .



The vector $\alpha\bar{u}$ is called the orthogonal projection of \bar{y} onto \bar{u} and is denoted $\text{proj}_{\bar{u}}\bar{y}$, where L is the 1-dimensional subspace spanned by \bar{u} .

Since $\bar{z} = \bar{y} - \hat{y} = \bar{y} - \alpha\bar{u}$ is orthogonal to \bar{u} ,

$$0 = (\bar{y} - \alpha\bar{u}) \cdot \bar{u} = \bar{y} \cdot \bar{u} - (\alpha\bar{u}) \cdot \bar{u} = \bar{y} \cdot \bar{u} - \alpha(\bar{u} \cdot \bar{u}).$$

So,

and

$$\alpha = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}$$

$$\hat{y} = \alpha\bar{u} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}$$

Example. Let $\bar{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of \bar{y} onto \bar{u} . Then write \bar{y} as the sum of two orthogonal vectors, one in $\text{Span}\{\bar{u}\}$ and one orthogonal to \bar{u} .

Solution. $\bar{y} \cdot \bar{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 28 + 12 = 40$

$$\bar{u} \cdot \bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 16 + 4 = 20$$

The orthogonal projection of \bar{y} onto \bar{u} is

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u} = \frac{40}{20} \bar{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix},$$

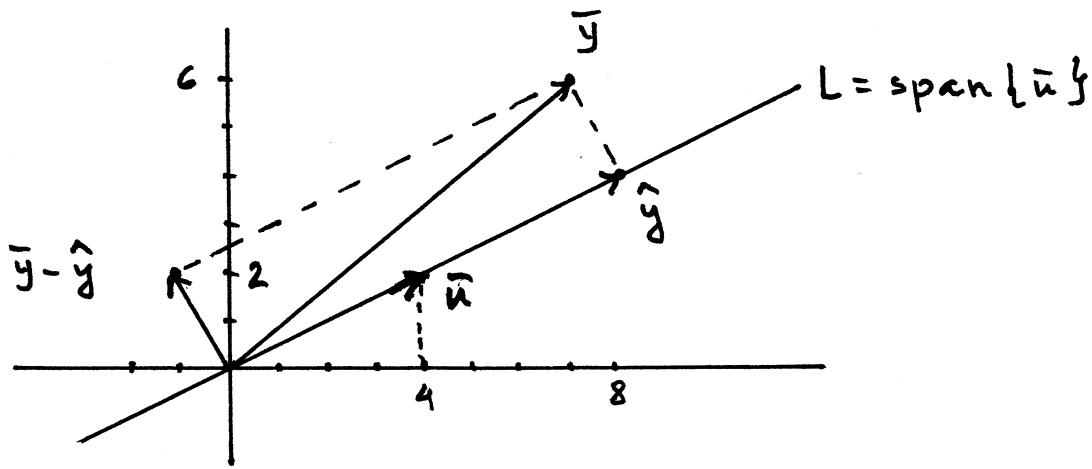
and the component of \bar{y} orthogonal to \bar{u} is

$$\bar{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

The decomposition is

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

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 \bar{y} \hat{y} $\bar{y} - \hat{y}$



Example. Find the distance from \bar{y} to L above.

Solution. The distance is $\|\bar{y} - \hat{y}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$.

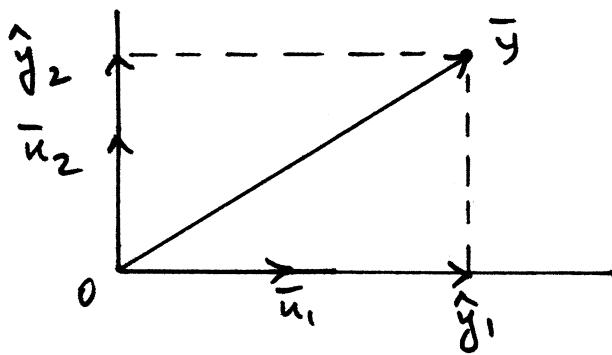
Geometric Interpretation of Orthogonal Projections

If \bar{u}_1, \bar{u}_2 is an orthogonal basis for \mathbb{R}^2 , then any vector \bar{y} in \mathbb{R}^2 can be written as

$$\bar{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} \bar{u}_2$$

where $\hat{y}_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} \bar{u}_1$ and $\hat{y}_2 = \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} \bar{u}_2$

are orthogonal projections of \bar{y} onto \bar{u}_1 and \bar{u}_2 , respectively.



Orthonormal Sets

Definition. A set $\{\bar{u}_1, \dots, \bar{u}_p\}$ of vectors in \mathbb{R}^n is called orthonormal if it is an orthogonal set of unit vectors.

The simplest example of an orthonormal set is the standard basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ for \mathbb{R}^n . Also, any nonempty subset of $\{\bar{e}_1, \dots, \bar{e}_n\}$ is orthonormal.

Example. Show that the vectors

$$\bar{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

form an orthonormal basis for \mathbb{R}^3 .

Solution. $\bar{v}_1 \cdot \bar{v}_2 = -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0$

$$\bar{v}_1 \cdot \bar{v}_3 = -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0$$

$$\bar{v}_2 \cdot \bar{v}_3 = 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0$$

Thus $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthogonal set. From

$$\bar{v}_1 \cdot \bar{v}_1 = 9/11 + 1/11 + 1/11 = 1$$

$$\bar{v}_2 \cdot \bar{v}_2 = 1/6 + 4/6 + 1/6 = 1$$

$$\bar{v}_3 \cdot \bar{v}_3 = 1/66 + 16/66 + 49/66 = 1$$

we conclude that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is orthonormal. Since an orthogonal set is linearly independent, it forms a basis for \mathbb{R}^3 .

Observation. If $\{\bar{v}_1, \dots, \bar{v}_p\}$ is an orthogonal set in \mathbb{R}^n , then the set $\{\frac{\bar{v}_1}{\|\bar{v}_1\|}, \dots, \frac{\bar{v}_p}{\|\bar{v}_p\|}\}$ is an orthonormal set.