

## 6.1 Inner Product, Length, and Orthogonality

Definition. Given vectors  $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$  and  $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  in  $\mathbb{R}^n$ ,

the product  $\bar{u}^T \bar{v}$ , also denoted  $\bar{u} \cdot \bar{v}$ , is called the dot product of  $\bar{u}$  and  $\bar{v}$ :

$$\bar{u}^T \bar{v} = \bar{u} \cdot \bar{v} = [u_1, u_2, \dots, u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n.$$

Example. Compute  $\bar{u} \cdot \bar{v}$ , where  $\bar{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$  and  $\bar{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$ .

Solution.  $\bar{u} \cdot \bar{v} = [2 \ -5 \ -1] \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = (2)(3) + (-5)(2) + (-1)(-3)$   
 $= -1.$

Theorem 1. Let  $\bar{u}, \bar{v}, \bar{w}$  be vectors in  $\mathbb{R}^n$ , and  $c$  be a scalar.

- a)  $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$
- b)  $(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$
- c)  $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v}) = \bar{u} \cdot (c\bar{v})$
- d)  $\bar{u} \cdot \bar{u} \geq 0$ , and  $\bar{u} \cdot \bar{u} = 0$  if and only if  $\bar{u} = \bar{0}$ .

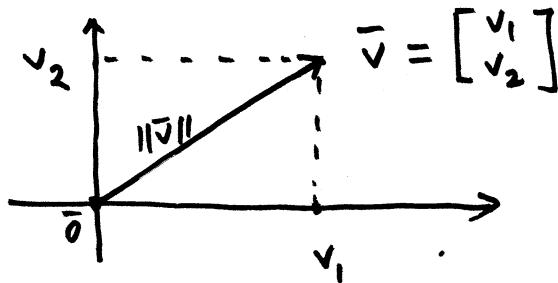
Properties b) and c) can be combined to produce the following rule:

$$(c_1 \bar{u}_1 + \dots + c_p \bar{u}_p) \cdot \bar{w} = c_1 (\bar{u}_1 \cdot \bar{w}) + \dots + c_p (\bar{u}_p \cdot \bar{w}).$$

## The Length of a Vector

Definition. The length (or norm) of a vector  $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  is the nonnegative scalar  $\|\bar{v}\|$  defined by

$$\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}.$$



Property. For any vector  $\bar{v}$  in  $R^n$  and any scalar  $c$ ,

$$\|c\bar{v}\| = |c| \|\bar{v}\|.$$

Definition. A vector  $\bar{u}$  in  $R^n$  whose length is 1 is called a unit vector.

Example.  $\bar{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$  is a unit vector in  $R^2$ .

Observation. If we divide a nonzero vector  $\bar{v}$  by its length—that is, multiply  $\bar{v}$  by  $\frac{1}{\|\bar{v}\|}$ —we obtain a unit vector  $\bar{u}$ , called a normal vector in the same direction as  $\bar{v}$ .

$$\bar{u} = \frac{1}{\|\bar{v}\|} \bar{v}, \text{ so } \|\bar{u}\| = \left\| \frac{\bar{v}}{\|\bar{v}\|} \right\| = \frac{1}{\|\bar{v}\|} \|\bar{v}\| = 1.$$

Example. Let  $\bar{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$ . Find a unit vector  $\bar{u}$  in the same direction as  $\bar{v}$ .

Solution.  $\|\bar{v}\|^2 = \bar{v} \cdot \bar{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$

$$\|\bar{v}\| = \sqrt{9} = 3.$$

$$\bar{u} = \frac{1}{\|\bar{v}\|} \bar{v} = \frac{1}{3} \bar{v} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

Indeed,  $\|\bar{u}\| = \left( \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 0^2 \right)^{1/2} = \left(\frac{1}{9} + \frac{4}{9} + \frac{4}{9}\right)^{1/2} = 1.$

### Distance in $R^n$

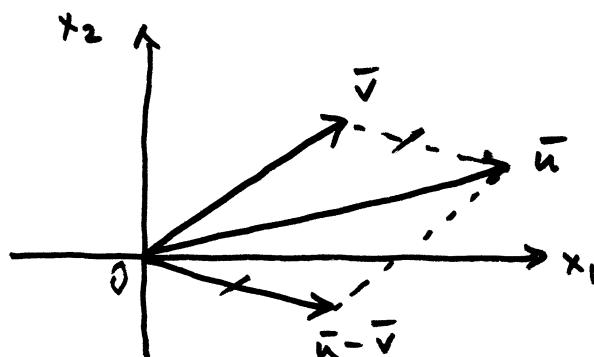
Definition. For  $\bar{u}$  and  $\bar{v}$  in  $R^n$ , the distance between  $\bar{u}$  and  $\bar{v}$ , written as  $\text{dist}(\bar{u}, \bar{v})$ , is the length of  $\bar{u} - \bar{v}$ :

$$\text{dist}(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|.$$

Example. Compute the distance between the vectors

$$\bar{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \text{ and } \bar{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

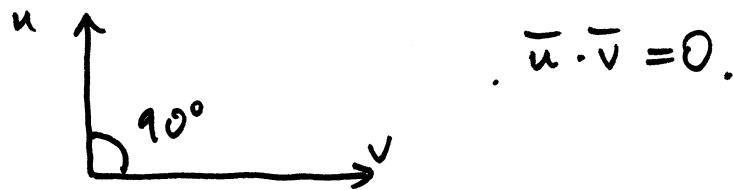
Solution.  $\bar{u} - \bar{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ ,  $\|\bar{u} - \bar{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$ .



## Orthogonal Vectors

Definition. Two vectors  $\bar{u}$  and  $\bar{v}$  in  $R^n$  are orthogonal (to each other) if  $\bar{u} \cdot \bar{v} = 0$ .

Observation. Vectors  $\bar{u}$  and  $\bar{v}$  in  $R^2$  or in  $R^3$  are orthogonal if and only if they form an angle of  $90^\circ$ .



Example The vectors  $\bar{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$  and  $\bar{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$  are orthogonal :  $\bar{u} \cdot \bar{v} = (2)(-3) + (6)(1) = 0$ .

## The Pythagorean Theorem.

Two vectors  $\bar{u}$  and  $\bar{v}$  in  $R^n$  are orthogonal if and only if  $\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$ .

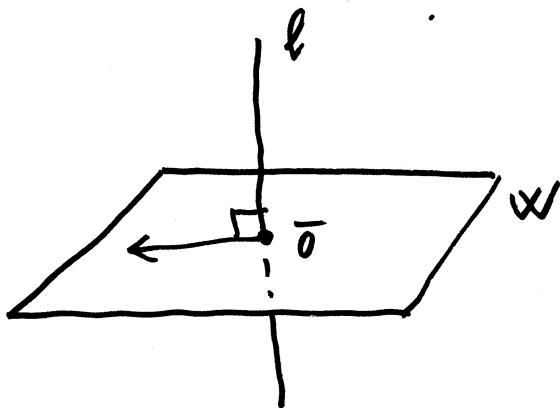
Proof.  $\|\bar{u} + \bar{v}\|^2 = (\bar{u} + \bar{v}) \cdot (\bar{u} + \bar{v}) = \bar{u} \cdot \bar{u} + \bar{u} \cdot \bar{v} + \bar{v} \cdot \bar{u} + \bar{v} \cdot \bar{v}$   
 $= \|\bar{u}\|^2 + 2 \bar{u} \cdot \bar{v} + \|\bar{v}\|^2$ .

so,  $\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$  if and only if  $\bar{u} \cdot \bar{v} = 0$ .

## Orthogonal Complements

Definition. If  $W$  is a subspace of  $\mathbb{R}^n$ , then we say that a vector  $\bar{z} \in \mathbb{R}^n$  is orthogonal to  $W$  provided  $\bar{z}$  is orthogonal to every vector in  $W$ .

The set of all vectors that are orthogonal to  $W$  is called the orthogonal complement of  $W$  and is denoted  $W^\perp$ .



Proposition. If  $W$  is a subspace of  $\mathbb{R}^n$ , then its orthogonal complement,  $W^\perp$ , is also a subspace of  $\mathbb{R}^n$ .

Proposition. If  $W$  is a subspace of  $\mathbb{R}^n$  and  $L = W^\perp$ , then  $W = L^\perp$ .

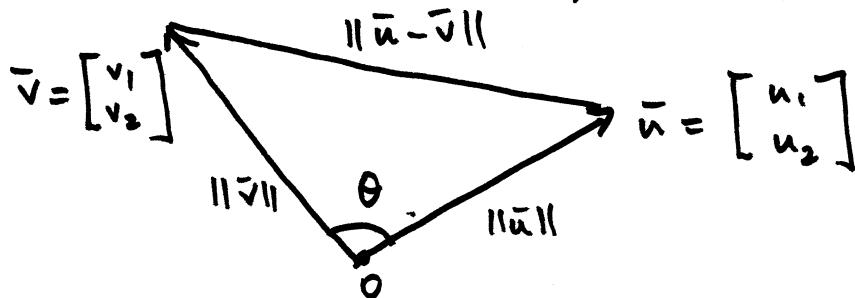
Theorem. Let  $A$  be an  $m \times n$  matrix. The row space of  $A$  and the column space of  $A$  have the following properties:  $(\text{Row } A)^\perp = \text{Nul } A$ ,  $(\text{Col } A)^\perp = \text{Nul}(A^T)$ .

## Angles in $R^2$ and $R^3$

Theorem If  $\bar{u}$  and  $\bar{v}$  are nonzero vectors in either  $R^2$  or  $R^3$  and  $\theta$  is the angle formed by  $\bar{u}$  and  $\bar{v}$ , then

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos \theta.$$

Proof. Let  $\bar{u}$  and  $\bar{v}$  belong to  $R^2$  (the case of  $R^3$  is similar). Put  $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ .



By the Law of Cosines,

$$\|\bar{u} - \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2 \|\bar{u}\| \|\bar{v}\| \cos \theta.$$

$$\begin{aligned} \text{Then } \|\bar{u}\| \|\bar{v}\| \cos \theta &= \frac{1}{2} (\|\bar{u}\|^2 + \|\bar{v}\|^2 - \|\bar{u} - \bar{v}\|^2) \\ &= \frac{1}{2} (u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2) \\ &= u_1 v_1 + u_2 v_2 = \bar{u} \cdot \bar{v}. \end{aligned}$$

So, the angle  $\theta$  between nonzero vectors  $\bar{u}$  and  $\bar{v}$  in  $R^n$  can be defined by

$$\cos \theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \|\bar{v}\|}, \quad 0 \leq \theta < \pi$$