## 5.3 Diagonalization

A square matrix $A$ is said to be <u>diagonalizable</u> if $A$ is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix $P$ and some diagonal matrix $D$.

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<u>Example 2.</u> The matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is diagonalizable. Indeed, let $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Then $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}.$$

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<u>Theorem 5</u> (The Diagonalization Theorem)

An $\underline{n} \times \underline{n}$ matrix $A$ is diagonalizable if and only if $A$ has $\underline{n}$ linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are $\underline{n}$ linearly independent eigenvectors of $A$ and the respective diagonal entries of $D$ are eigenvalues of $A$.

## Diagonalizing Matrices

**Example 3.** Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

**Solution.**

**Step 1.** Find the eigenvalues of $A$.

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2.$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

**Step 2.** Find three linearly independent eigenvectors of $A$. Solving the matrix equations

$$(A - I)\bar{x} = \bar{0} \qquad \text{and} \qquad (A + 2I)\bar{x} = \bar{0},$$

we find bases for each eigenspace:

Basis for $\lambda = 1$: $\quad \bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Basis for $\lambda = -2$: $\quad \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\bar{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

Since the matrix

$$\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

is invertible, the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly independent.

Step 3. Construct P from the vectors in Step 2.

$$P = [\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is easy to see that

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Step 4. Construct D from the corresponding eigenvalues:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Step 5. Write the equation $A = PDP^{-1}$.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = PDP^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

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Example 4. Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution. As in Example 3,

$$0 = \det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2.$$

So, the eigenvalues of A are $\lambda = 1$ and $\lambda = -2$.

However, both eigenspaces are 1-dimensional:

Basis for $\lambda=1$ is $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; for $\lambda_2=-2$ - $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

It is impossible to construct a basis for $R^3$ using eigenvectors of A. By Theorem 5, A is not diagonalizable.

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Theorem 6. An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

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Matrices Whose Eigenvalues are not Distinct

Theorem 7. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. The dimension of the eigenspace for $\lambda_k$, $1 \le k \le p$, is less than or equal to the multiplicity of $\lambda_k$.

b. The matrix A is diagonalizable if and only if the sum of dimensions of eigenspaces equals $n$.

c. A is diagonalizable if and only if (i) the characteristic polynomial of A factors into linear polynomials, (ii) the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

d. If $A$ is diagonalizable and $B_k$ is a basis for the eigenspace corresponding to $\lambda_k$, $1 \le k \le p$, then the total collection of vectors in the sets $B_1, \dots, B_p$ forms an eigenvector basis for $R^n$.

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Example 6. Diagonalize the matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Solution. Since the matrix is triangular, its eigenvalues are 5 and $-3$, each of multiplicity 2. Using the method in Section 5.1, we find a basis for each eigenspace:

For $\lambda = 5$: $\bar{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$, for $\lambda = -3$: $\bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

So, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$