

5.2. The Characteristic Equation

If \bar{x} is an eigenvector of an $n \times n$ matrix A , then $A\bar{x} = \lambda\bar{x}$, or $(A - \lambda I)\bar{x} = \bar{0}$. Hence $A - \lambda I$ is not invertible, which gives that $\det(A - \lambda I) = 0$.

Definition. The scalar equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A .

Example. Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution. We must find all scalars λ such that the matrix equation $(A - \lambda I)\bar{x} = \bar{0}$ has a nontrivial solution, or such that $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}.$$

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-6-\lambda) - (3)(3) \\ &= -12 + 6\lambda - 2\lambda + \lambda^2 - 9 = \lambda^2 + 4\lambda - 21. \end{aligned}$$

Setting $\lambda^2 + 4\lambda - 21 = 0$, we have $(\lambda-3)(\lambda+7)=0$. So, the eigenvalues of A are 3 and -7.

Example. Find the eigenvalues of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)^2(3-\lambda)(1-\lambda) = 0.$$

So, $\lambda = 5$, $\lambda = 3$, and $\lambda = 1$ are the eigenvalues of A.
