

5.1. Eigenvectors and Eigenvalues

Definition. An eigenvector of an $n \times n$ matrix A is a nonzero vector \bar{x} such that $A\bar{x} = \lambda\bar{x}$ for some scalar λ . A scalar λ is called an eigenvalue of A if there is a nontrivial solution \bar{x} of $A\bar{x} = \lambda\bar{x}$; such an \bar{x} is called an eigenvector corresponding to λ .

Example. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\bar{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Then

$$A\bar{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\bar{u},$$

which shows that \bar{u} is an eigenvector of A . Similarly,

$$A\bar{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ shows that } \bar{v} \text{ is not$$

an eigenvector of A .

If \bar{x} is an eigenvector of A , that is, $A\bar{x} = \lambda\bar{x}$ for some scalar λ , then we can write

$$(A - \lambda I)\bar{x} = \bar{0}.$$

The set of all solutions of this equation is a subspace of \mathbb{R}^n and is called the eigenspace of A corresponding to λ .

Example. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis for the eigenspace that corresponds to $\lambda = 2$.

Solution. $A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$.

Since

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the general solution of $(A - 2I)\bar{x} = \bar{0}$ is given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2/2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the eigenspace.

Theorem 1. The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example. If $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 7 & 5 \\ 0 & 0 & 2 \end{bmatrix}$, then $\lambda_1 = 3$, $\lambda_2 = 7$, and $\lambda_3 = 2$ are the eigenvalues of A .

Theorem 2. If $\bar{v}_1, \dots, \bar{v}_p$ are eigenvectors of A that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_p$, then the set $\{\bar{v}_1, \dots, \bar{v}_p\}$ is linearly independent.