

4.7. Change of Basis

In some applications, a problem is described initially using a basis B , but the problem's solution is aided by changing B to a new basis C .

Example 1. Consider two bases $B = \{\bar{b}_1, \bar{b}_2\}$ and $C = \{\bar{c}_1, \bar{c}_2\}$ for $\mathbb{R}^2$, such that

$$\bar{b}_1 = 4\bar{c}_1 + \bar{c}_2, \quad \bar{b}_2 = -6\bar{c}_1 + \bar{c}_2.$$

Suppose $\bar{x} = 3\bar{b}_1 + \bar{b}_2$; that is $[\bar{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (see Section 2.9). Find $[\bar{x}]_C$.

Solution.
$$[\bar{x}]_C = [3\bar{b}_1 + \bar{b}_2]_C = 3[\bar{b}_1]_C + [\bar{b}_2]_C$$
$$= 3 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-6 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Another way:

$$[\bar{x}]_C = 3[\bar{b}_1]_C + 1[\bar{b}_2]_C = \begin{bmatrix} [\bar{b}_1]_C & [\bar{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Theorem 15.

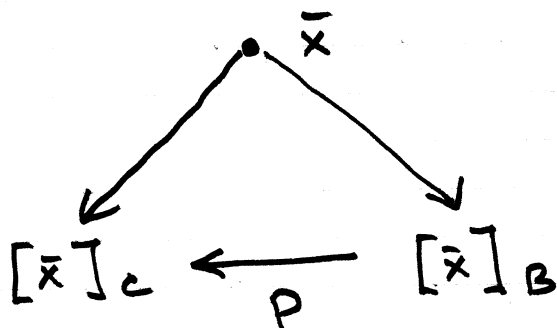
Let $B = \{\bar{b}_1, \dots, \bar{b}_n\}$ and $C = \{\bar{c}_1, \dots, \bar{c}_n\}$ be two bases for $\mathbb{R}^n$. Then there is a unique $n \times n$ matrix P (also denoted $c \leftarrow B$) such that

$$[\bar{x}]_C = P [\bar{x}]_B \text{ for any vector } \bar{x} \text{ in } \mathbb{R}^n.$$

Furthermore, the columns of P are the C -coordinate vectors of $\bar{b}_1, \dots, \bar{b}_n$ respectively:

$$P = [[\bar{b}_1]_C \dots [\bar{b}_n]_C].$$

The matrix P in Theorem 15 is called the change-of-coordinate matrix from B to C .



The columns of P are linearly independent; so, the matrix P is invertible due to the Invertible Matrix Theorem (see Section 2.9).

Therefore, $[\bar{x}]_B = P^{-1} [\bar{x}]_C.$

Computing the matrix P.

Example 2. Let $\bar{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\bar{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\bar{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\bar{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Compute the change-of-coordinates matrix P from B to C, where $B = \{\bar{b}_1, \bar{b}_2\}$ and $C = \{\bar{c}_1, \bar{c}_2\}$.

Solution. Form the matrix $[\bar{c}_1, \bar{c}_2, \bar{b}_1, \bar{b}_2]$ and transform it into the reduced echelon form:

$$[\bar{c}_1, \bar{c}_2, \bar{b}_1, \bar{b}_2] = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}.$$

Then $P = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$.

Remark. The change-of-coordinates matrix from C to B above is P^{-1} :

$$P^{-1} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -3/2 & 2 \\ 5/2 & 3 \end{bmatrix}.$$