4.7. Change of Basis

In some applications, a problem is described initially using a basis B, but the problem's solution is aided by changing B to a new basis C.

__Example 1.__ Consider two bases $B = \{\bar{b}_1, \bar{b}_2\}$ and $C = \{\bar{c}_1, \bar{c}_2\}$ for R^2, such that

$$\bar{b}_1 = 4\bar{c}_1 + c_2, \quad \bar{b}_2 = -6\bar{c}_1 + \bar{c}_2.$$

Suppose $\bar{x} = 3\bar{b}_1 + \bar{b}_2$; that is $[\bar{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (see Section 2.9). Find $[\bar{x}]_C$.

__Solution.__ $[\bar{x}]_C = [3\bar{b}_1 + \bar{b}_2]_C = 3[\bar{b}_1]_C + [\bar{b}_2]_C$

$$= 3\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12-6 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Another way:

$$[\bar{x}]_C = 3[\bar{b}_1]_C + 1[\bar{b}_2]_C = \begin{bmatrix} [\bar{b}_1]_C & [\bar{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Theorem 15.

Let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ and $C = \{\bar{c}_1, \ldots, \bar{c}_n\}$ be two bases for R^n. Then there is a unique $n \times n$ matrix P (also denoted $\underset{C \leftarrow B}{P}$) such that

$$[\bar{x}]_C = P[\bar{x}]_B \text{ for any vector } \bar{x} \text{ in } R^n.$$

Furthermore, the columns of P are the C-coordinate vectors of $\bar{b}_1, \ldots, \bar{b}_n$ respectively:

$$P = \big[\, [\bar{b}_1]_C \ \ldots \ [\bar{b}_n]_C \,\big].$$

The matrix P in Theorem 15 is called the change-of-coordinate matrix from B to C.

$$\begin{tikzcd}
 & \bar{x} \arrow[dl] \arrow[dr] & \\
{[\bar{x}]_C} & & {[\bar{x}]_B} \arrow[ll, "P"']
\end{tikzcd}$$

The columns of P are linearly independent; so, the matrix P is invertible due to the Invertible Matrix Theorem (see Section 2.9).

Therefor, $[\bar{x}]_B = P^{-1}[\bar{x}]_C$.

Computing the matrix P.

Example 2. Let $\bar{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\bar{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\bar{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\bar{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.
Compute the change-of-coordinates matrix P from B to C, where $B = \{\bar{b}_1, \bar{b}_2\}$ and $C = \{\bar{c}_1, \bar{c}_2\}$.

Solution. Form the matrix $[\bar{c}_1\ \bar{c}_2\ \bar{b}_1\ \bar{b}_2]$ and transform it into the reduced echelon form:

$$[\bar{c}_1\ \bar{c}_2\ \bar{b}_1\ \bar{b}_2] = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}.$$

Then $P = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$.

Remark. The change-of-coordinates matrix from C to B above is P^{-1}:

$$P^{-1} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} -3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -3/2 & 2 \\ 5/2 & 3 \end{bmatrix}.$$