

## 4.4. Coordinate Systems

Theorem 7. Let  $B = \{\bar{b}_1, \dots, \bar{b}_n\}$  be a basis for the vector space  $\mathbb{R}^n$ . Then each vector  $\bar{x}$  in  $\mathbb{R}^n$  is uniquely expressible as a linear combination

$$\bar{x} = c_1 \bar{b}_1 + \dots + c_n \bar{b}_n.$$

Definition. Suppose  $B = \{\bar{b}_1, \dots, \bar{b}_n\}$  is a basis for  $\mathbb{R}^n$  and  $\bar{x}$  is in  $\mathbb{R}^n$ . The coordinates of  $\bar{x}$  relative to the basis  $B$  (or the  $B$ -coordinates of  $\bar{x}$ ) are the (unique) coefficients  $c_1, \dots, c_n$  such that

$$\bar{x} = c_1 \bar{b}_1 + \dots + c_n \bar{b}_n.$$

Notation. If  $c_1, \dots, c_n$  are the  $B$ -coordinates of  $\bar{x}$ , then the vector

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of  $\bar{x}$ , or the  $B$ -coordinate vector  $\bar{x}$ .

The mapping  $\bar{x} \mapsto [\bar{x}]_B$  is called the coordinate mapping (determined by  $B$ ).

Example 1. Consider the basis  $B = \{\bar{b}_1, \bar{b}_2\}$  for  $\mathbb{R}^2$ :

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find  $\bar{x}$  if  $[\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ .

Solution.

$$\bar{x} = (-2)\bar{b}_1 + 3\bar{b}_2 = (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Example 2. If  $E = \{\bar{e}_1, \bar{e}_2\}$  is the standard basis for  $\mathbb{R}^2$ , then for any vector  $\bar{x}$  in  $\mathbb{R}^n$ ,

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which shows that  $\bar{x} = [\bar{x}]_E$ .

Example 4. Let  $\bar{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\bar{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $\bar{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ . If  $B = \{\bar{b}_1, \bar{b}_2\}$  find  $[\bar{x}]_B$ .

Solution. The  $B$ -coordinates  $c_1, c_2$  of  $\bar{x}$  satisfy

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Solving this vector equation for  $c_1$  and  $c_2$ , we get

$$c_1 = 3, c_2 = 2. \text{ So } [\bar{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Definition. Let  $B = \{\bar{b}_1, \dots, \bar{b}_n\}$  be a basis for  $\mathbb{R}^n$ .

The matrix  $P_B = [\bar{b}_1 \dots \bar{b}_n]$  is called the change-of-coordinates matrix from  $B$  to the standard basis  $E = \{\bar{e}_1, \dots, \bar{e}_n\}$  in  $\mathbb{R}^n$ .

If  $\bar{x} = c_1 \bar{b}_1 + \dots + c_n \bar{b}_n$ , then

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = P_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P_B [\bar{x}]_B.$$

Since the columns of  $P_B$  form a basis for  $\mathbb{R}^n$ , the Invertible Matrix Theorem shows that  $P_B$  is invertible. Therefore,

$$[\bar{x}]_B = P_B^{-1} \bar{x}.$$

Theorem 8. Let  $B = \{\bar{b}_1, \dots, \bar{b}_n\}$  be a basis for  $\mathbb{R}^n$ .

The coordinate mapping  $\bar{x} \rightarrow [\bar{x}]_B$  is a 1-1 linear transformation from  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .