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3.1. Introduction to Determinants

We recall from Section 2.2 that a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if its determinant,

$$\det A = a_{11}a_{22} - a_{12}a_{21},$$

is not zero.

Our goal is to generalize this statement to the case of square matrices of size $n \times n$, where $n \geq 1$.

If $n=1$, then $A = [a_{11}]$, and we let $\det A = a_{11}$.

The case $n=2$ is described above.

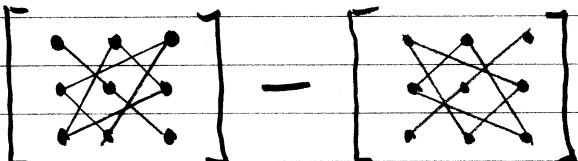
Let $n=3$. Then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We let

$$\begin{aligned} \det A = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

The above expression can be graphically interpreted as follows:



One can easily see that the determinant of a 3×3 matrix can be reduced to a computation of determinants of 2×2 matrices:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Example 1. Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Solution.

$$\begin{aligned} \det A &= 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1(0-2) - 5(0-0) + 0(-4-0) = -2. \end{aligned}$$

We now give a recursive definition of determinant.

Definition. For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}, \end{aligned}$$

where A_{ij} denotes the $(n-1) \times (n-1)$ matrix obtained from A by deleting row 1 and column j .

For instance, if A is a 3×3 matrix, then

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

Given an $n \times n$ matrix A , the (i, j) -cofactor of A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij},$$

where A_{ij} is the submatrix of A obtained from A by deleting row i and column j .

The formula in the definition of $\det A$ deals with a cofactor expansion across the first row of A and can be written as

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}.$$

Theorem 1. The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row and down any column:

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in},$$

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}.$$

(4)

The distribution of plus or minus sign in the (i,j) -cofactor expansion, generated by $(-1)^{i+j}$, has a "chess" configuration:

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \dots$$

Example 2. Use a cofactor expansion across the third row to compute $\det A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

$$\begin{aligned} \text{Solution. } \det A &= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} \\ &= (-1)^{3+1} a_{31} \det A_{31} + (-1)^{3+2} a_{32} \det A_{32} + (-1)^{3+3} a_{33} \det A_{33} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2. \end{aligned}$$

Special notation: If A is an $n \times n$ matrix, then we put $\det A = |A|$.

Example 3. Compute $\det A$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

Solution.

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 3 \cdot 2 \cdot 1 \cdot \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot (-2) = -12$$

Theorem 2. If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A .