

## 2.9. Dimension and Rank

The main reason for selecting a basis for a subspace  $H$  is that every vector in  $H$  can be written in only one way as a linear combination of the basis vectors. To see why, suppose  $B = \{\bar{b}_1, \dots, \bar{b}_p\}$  is a basis for  $H$ , and let  $\bar{x}$  be a vector in  $H$  generated in two ways, say

$$\bar{x} = c_1 \bar{b}_1 + \dots + c_p \bar{b}_p \quad \text{and} \quad \bar{x} = d_1 \bar{b}_1 + \dots + d_p \bar{b}_p.$$

Subtraction gives

$$\bar{0} = \bar{x} - \bar{x} = (c_1 - d_1) \bar{b}_1 + \dots + (c_p - d_p) \bar{b}_p.$$

Since  $\bar{b}_1, \dots, \bar{b}_p$  are linearly independent,

$$c_1 = d_1, \dots, c_p = d_p.$$

Def. Suppose the set  $B = \{\bar{b}_1, \dots, \bar{b}_p\}$  is a basis for a subspace  $H$ . For each  $\bar{x}$  in  $H$ , the coordinates of  $\bar{x}$  relative to the basis  $B$  are the scalars  $c_1, \dots, c_p$  such that  $\bar{x} = c_1 \bar{b}_1 + \dots + c_p \bar{b}_p$ , and the vector in  $\mathbb{R}^p$

$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$  is called the coordinate vector of  $\bar{x}$  or the  $B$ -coordinate vector of  $\bar{x}$ .

Ex. Let  $\bar{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ ,  $\bar{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\bar{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ , and  $B = \{\bar{b}_1, \bar{b}_2\}$ .

Then  $B$  is a basis for  $H = \text{Span}\{\bar{v}_1, \bar{v}_2\}$ . It is easy to see that  $\bar{x} = 2\bar{v}_1 + 3\bar{v}_2$ , so  $[\bar{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

In general, if  $B = \{\bar{b}_1, \dots, \bar{b}_p\}$  is a basis for  $H$ , then the mapping  $\bar{x} \mapsto [\bar{x}]_B$  is a one-to-one correspondence (called isomorphism) that makes  $H$  look and act the same as  $\mathbb{R}^p$ , even though the vectors in  $H$  belong to  $\mathbb{R}^n$ .

### The Dimension of a Subspace

It can be shown that if a subspace  $H$  has a basis of  $p$  vectors, then every basis of  $H$  must consist of exactly  $p$  vectors.

Definition. The dimension of a nonzero subspace  $H$ , denoted by  $\dim H$ , is the number of vectors in any basis for  $H$ . The dimension of the zero subspace  $\{\bar{0}\}$  is defined to be zero.

Remark. The zero subspace  $\{\bar{0}\}$  has no basis because the zero vector  $\bar{0}$  by itself forms a linearly dependent set.

The space  $\mathbb{R}^n$  has dimension  $n$ , and every basis for  $\mathbb{R}^n$  consists of  $n$  vectors.

Def. The rank of a matrix  $A$ , denoted  $\text{rank } A$ , is the dimension of the column space of  $A$ .

Theorem (The Rank Theorem).

If a matrix  $A$  has  $n$  columns, then

$$\text{rank } A + \dim \text{Nul } A = n.$$


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Indeed,  $\text{rank } A$  equals the number of pivot columns of  $A$ , while  $\dim \text{Nul } A$  equals the number of free variables of the equation  $A\bar{x} = \bar{0}$ .

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The Basis Theorem:

Let  $H$  be a  $p$ -dimensional subspace of  $\mathbb{R}^n$ . Any linearly independent set of exactly  $p$  elements in  $H$  is automatically a basis for  $H$ . Also, any set of  $p$  elements of  $H$  that spans  $H$  is automatically a basis for  $H$ .

## The Invertible Matrix Theorem (continued)

Let  $A$  be an  $n \times n$  matrix. Then the following statements are each equivalent to the statement that  $A$  is an invertible matrix:

- m) the columns of  $A$  form a basis for  $\mathbb{R}^n$ ,
- n)  $\text{Col } A = \mathbb{R}^n$ ,
- o)  $\dim \text{Col } A = n$ ,
- p)  $\text{rank } A = n$ ,
- q)  $\text{Nul } A = \{\vec{0}\}$ ,
- r)  $\dim \text{Nul } A = 0$ .