

2.8. Subspaces of R^n

Definition. A subspace of R^n is any set H in R^n that has three properties:

- a) the zero vector $\vec{0}$ is in H ,
- b) for any \vec{u} and \vec{v} in H , the sum $\vec{u} + \vec{v}$ is in H ,
- c) for any \vec{u} in H and any scalar c , the vector $c\vec{u}$ is in H .

In words, a subspace is closed under addition and scalar multiplication.

Ex. Given vectors $\vec{v}_1, \dots, \vec{v}_p$ in R^n , the span $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace.

Proof. a) The zero vector $\vec{0}$ can be written as

$$\vec{0} = 0\vec{v}_1 + \dots + 0\vec{v}_p, \text{ so } \vec{0} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}.$$

- b) If \vec{u} and \vec{v} are any vectors in $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$, then $\vec{u} = \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p$, $\vec{v} = \beta_1\vec{v}_1 + \dots + \beta_p\vec{v}_p$, and $\vec{u} + \vec{v} = (\alpha_1 + \beta_1)\vec{v}_1 + \dots + (\alpha_p + \beta_p)\vec{v}_p \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$.
- c) If \vec{u} is in $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$, then $\vec{u} = \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p$, and, for any scalar c , $c\vec{u} = c\alpha_1\vec{v}_1 + \dots + c\alpha_p\vec{v}_p \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$.

Definition. The column space of a matrix A is the set $\text{Col } A$ of all linear combinations of the columns of A .

If a matrix A is given as $A = [\bar{a}_1 \dots \bar{a}_n]$, with columns $\bar{a}_1, \dots, \bar{a}_n$ in \mathbb{R}^m , then

$$\text{Col } A = \text{Span}\{\bar{a}_1, \dots, \bar{a}_n\}.$$

By the above, $\text{Col } A$ is a subspace of \mathbb{R}^m .

Ex. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

Determine whether \bar{b} is in $\text{Col } A$.

Solution. The vector \bar{b} is in $\text{Col } A$ if and only if the equation $A\bar{x} = \bar{b}$ has a solution. Row reducing the augmented matrix $[A \bar{b}]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that $A\bar{x} = \bar{b}$ is consistent and \bar{b} is in $\text{Col } A$.

Definition. The null space of a matrix A is the set $\text{Nul } A$ of all solutions to the homogeneous equation $A\bar{x} = \bar{0}$.

Theorem 12. The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n . Equivalently, the set of all solutions to a system $A\bar{x} = \bar{0}$ of m homogeneous linear equations in n unknowns is a subspace of \mathbb{R}^n .

Proof. a) The zero vector $\bar{0}$ is in $\text{Nul } A$ because of $A\bar{0} = \bar{0}$.

b) If \bar{u} and \bar{v} are in $\text{Nul } A$, that is, $A\bar{u} = \bar{0}$ and $A\bar{v} = \bar{0}$, then $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = \bar{0} + \bar{0} = \bar{0}$. Hence $\bar{u} + \bar{v}$ is in $\text{Nul } A$.

c) If \bar{u} is in $\text{Nul } A$ and c is a scalar, then $A(c\bar{u}) = cA\bar{u} = c\bar{0} = \bar{0}$ shows that $c\bar{u}$ is in $\text{Nul } A$. Summing up, $\text{Nul } A$ is a subspace.

Basis for a Subspace

Definition. A basis for a subspace H of \mathbb{R}^n is a linearly independent set in H that spans H .

Ex. The columns $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $\bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ form a basis for \mathbb{R}^n .

Ex. The columns of an invertible $n \times n$ matrix form a basis for \mathbb{R}^n because they are linearly independent and span \mathbb{R}^n , by the Invertible Matrix Theorem.

The next example shows that the standard procedure for writing the solution set of the equation $A\bar{x} = \bar{0}$ in parametric vector form actually identifies a basis for $\text{Nul } A$.

Ex. Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. First, write the solution of $A\bar{x} = \bar{0}$ in parametric form:

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

x_2, x_4 , and x_5 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow \bar{u} \quad \uparrow \bar{v} \quad \uparrow \bar{w}$

$$= x_2 \bar{u} + x_4 \bar{v} + x_5 \bar{w}.$$

So, $\text{Nul } A = \text{Span}\{\bar{u}, \bar{v}, \bar{w}\}$. It is easy to see that the equality $\bar{0} = x_2 \bar{u} + x_4 \bar{v} + x_5 \bar{w}$ holds only if $x_2 = x_4 = x_5$, which means that $\bar{u}, \bar{v}, \bar{w}$ are linearly independent. Thus, $\bar{u}, \bar{v}, \bar{w}$ is a basis for $\text{Nul } A$.

How to find a basis for the column space of a matrix?

Ex. Find a basis for the column space of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. Denote by $\bar{b}_1, \dots, \bar{b}_5$ the columns of B . Then $\text{Col } B = \text{Span}\{\bar{b}_1, \dots, \bar{b}_5\}$. Observe that

$$\bar{b}_3 = -3\bar{b}_1 + 2\bar{b}_2 \text{ and } \bar{b}_4 = 5\bar{b}_1 - \bar{b}_2.$$

So, any vector \bar{v} in $\text{Col } B$ can be written as

$$\begin{aligned}\bar{v} &= c_1 \bar{b}_1 + c_2 \bar{b}_2 + c_3 \bar{b}_3 + c_4 \bar{b}_4 + c_5 \bar{b}_5 \\ &= c_1 \bar{b}_1 + c_2 \bar{b}_2 + c_3 (-3\bar{b}_1 + 2\bar{b}_2) + c_4 (5\bar{b}_1 - \bar{b}_2) + c_5 \bar{b}_5 \\ &= (c_1 - 3c_3 + 5c_4)\bar{b}_1 + (c_2 + 2c_3 - c_4)\bar{b}_2 + c_5 \bar{b}_5\end{aligned}$$

So, $\text{Col } B = \text{Span}\{\bar{b}_1, \bar{b}_2, \bar{b}_5\}$. Since $\bar{b}_1, \bar{b}_2, \bar{b}_5$ are linearly independent, they form a basis for $\text{Col } B$.

In the example above, B is in reduced echelon form. How to handle the case of a general matrix?

Ex. Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & \neq & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}.$$

Solution. Denote by $\bar{a}_1, \dots, \bar{a}_5$ the columns of A . It can be verified that A is row equivalent to the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the previous example we know that $\text{Col } B = \text{Span}\{\bar{b}_1, \bar{b}_2, \bar{b}_5\}$, and that $\bar{b}_3 = -3\bar{b}_1 + 2\bar{b}_2$ and $\bar{b}_4 = 5\bar{b}_1 - \bar{b}_2$. It is possible to show that row operations do not affect linear dependence relations among the columns

of the matrix. So, $\bar{a}_1, \bar{a}_2, \bar{a}_5$ are linearly independent and $\bar{a}_3 = -3\bar{a}_1 + 2\bar{a}_2$, $\bar{a}_4 = 5\bar{a}_1 - \bar{a}_2$. Thus $\bar{a}_1, \bar{a}_2, \bar{a}_5$ is a basis for $\text{Col } A$.

Warning: A basis for $\text{Col } A$ is formed by the pivot columns of the original matrix A and not by the pivot columns of its echelon form B .

Theorem. The pivot columns of a matrix A form a basis for the column space of A .