2.3 Characterizations of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)

If A is a square $n \times n$ matrix, then the following statements are equivalent.

a. A is invertible.

b. A is row equivalent to the identity matrix I_n.

c. A has n pivot positions.

d. The equation $A\bar{x} = \bar{0}$ has only trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\bar{x} \to A\bar{x}$ is one-to-one.

g. The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

h. The columns of A span R^n.

i. The linear transformation $\bar{x} \to A\bar{x}$ maps R^n onto R^n.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Corollary. If A and B are square matrices such that $AB = I$, then both A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Example 1. Use the Invertible Matrix Theorem to decide if the matrix A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}.$$

Solution. $A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

By any of the statements (b) and (c), A is invertible.

Invertible Linear Transformations

A linear transformation $T: R^n \to R^n$ is said to be *invertible* if there exists a mapping $S: R^n \to R^n$ such that

$$S(T(\bar{x})) = \bar{x} \text{ and } T(S(\bar{x})) = \bar{x} \text{ for all } \bar{x} \text{ in } R^n.$$

Theorem 9. Let $T: R^n \to R^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only A is invertible. In that case, the linear transformation $S(\bar{x}) = A^{-1}\bar{x}$ is the inverse to T transformation.

Indeed, $S(T(\bar{x})) = A^{-1}(A\bar{x}) = (A^{-1}A)\bar{x} = I\bar{x} = \bar{x}$ for all $\bar{x}$ in R^n.