

2.2. The Inverse of a Matrix

Definition. An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I, \quad (*)$$

where $I = I_n$ is the $n \times n$ identity matrix. A non-invertible matrix is called a singular matrix.

Statement. The matrix C in $(*)$ is uniquely determined by A . Indeed, if B were another matrix satisfying the condition $BA = AB = I$, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

This unique matrix C is called the inverse of A and is denoted A^{-1} .

Example 1. If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = CA = I. \text{ So, } C = A^{-1}.$$

Theorem 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

The quantity $ad - bc$ is called the determinant of A , and we write $\det A = ad - bc$.

Example 2. Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Solution. Since $\det A = 3 \cdot 6 - 4 \cdot 5 = -2 \neq 0$, A is invertible and $A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$.

Theorem 5. If A is an invertible $n \times n$ matrix, then for each \bar{b} in R^n , the equation $A\bar{x} = \bar{b}$ has the unique solution $\bar{x} = A^{-1}\bar{b}$.

Indeed, if $A\bar{x} = b$, then

$$\bar{x} = I\bar{x} = (A^{-1}A)\bar{x} = A^{-1}(A\bar{x}) = A^{-1}\bar{b}.$$

Example 4. Use the inverse of the matrix A in Example 2 to solve the system

$$3x_1 + 4x_2 = 3$$

$$5x_1 + 6x_2 = 7.$$

Solution. This system is equivalent to $A\bar{x} = \bar{b}$; so

$$\bar{x} = A^{-1}\bar{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Theorem 6. Let A and B be $n \times n$ matrices. Then

a. If A is invertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

b. If both A and B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

c. If A is invertible, then so is A^T and

$$(A^T)^{-1} = (A^{-1})^T.$$

An Algorithm for Finding A^{-1}

Row reduce the augmented matrix $[AI]$. If A is row equivalent to I, then $[AI]$ is row equivalent to $[IA^{-1}]$. Otherwise, A does not have an inverse.

Example 7. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$.

Solution.

$$[AI] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.$$

So, A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$